Solution of Mid-Term Test.

1. Proof (i) ① By the definition of ℓ^1 norm,

$$\|M_x(y)\|_{\ell^1} = \sum_{k=1}^{\infty} |x(k)y(k)| \leq \|x\|_{\ell^\infty} \sum_{k=1}^{\infty} |y(k)|$$
$$= \|x\|_{\ell^\infty} \|y\|_{\ell^1}$$

$$\Rightarrow \|M_x\| \leq \|x\|_{\ell^\infty}$$

② The definiton of ℓ^∞ norm implies that

there exists $\{k_n\}_{n=1}^{\infty}$ s.t. $\|x\|_{\ell^\infty} - \frac{1}{n} < |x(k_n)| \leq \|x\|_{\ell^\infty}$

Now consider the sequces $e_{k_n} = \begin{cases} 1 & k = k_n \\ 0 & \text{otherwise.} \end{cases}$

Then $\|e_{k_n}\|_{\ell^1} = 1$, $\|M_x(e_{k_n})\|_{\ell^1} = |x(k_n)| \to \|x\|_{\ell^\infty}$ as $n \to \infty$.

Therefore $\|M_x\| \geq \|x\|_{\ell^\infty}$.

(ii) For any $z \in (\ell^1)^* = \ell^\infty$, $z(y) = \sum_{k=1}^{\infty} z(k)y(k)$, $\forall y \in \ell^1$.

Then $\left(M_x^* z\right)(y) = z(M_x y) = \sum_{k=1}^{\infty} z(k)\left[x(k)\, y(k)\right]$

$$= \sum_{k=1}^{\infty} \left[x(k)\, z(k)\right] y(k) \qquad \forall y \in \ell^1$$

Therefore M_x^* is defined as $M_x^* : \ell^\infty \to \ell^\infty$

$z \mapsto M_x^*(z)$

$M_x^*(z)(k) = x(k)z(k)$

□

2. Sol. (i) ① Since $\left|\int_a^x f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt = \|f\|_1$,

we have $\|T\| \leq 1$.

② Meanwhile, define $f(t) = \frac{1}{b-a}$ and then $\|f\|_1 = 1$.

$|Tf(x)| = \frac{x-a}{b-a} \to 1$ as $x \to b$. So $\|Tf\|_\infty = 1$.

And thus, $\|T\| \geq 1$.

Combining ① & ② gives $\|T\| = 1$.

(ii) ① $\|Tf\|_1 = \int_a^b \left|\int_a^x f(t)\,dt\right| dx \leq \int_a^b \|f\|_1\,dx$

$\leq (b-a)\|f\|_1$

② Define $f_n(t) = \begin{cases} n^2(x-a) & a \leq t \leq a+\frac{1}{n} \\ -n^2(x-a)+2n & a+\frac{1}{n} < t \leq a+\frac{2}{n} \\ a & \text{otherwise} \end{cases}$.

Where $n \in \mathbb{N}^*$, and $a+\frac{2}{n} \leq b$.

Therefore, $f_n \in X$, $\|f_n\|_1 = 1$, $\int_a^x f_n(t)\,dt = 1$ when $x \geq a+\frac{2}{n}$.

And thus $\|Tf_n\|_1 \geq \int_{a+\frac{2}{n}}^b 1\,dx = \left(b-a-\frac{2}{n}\right)$

$\to b-a$ as $n \to \infty$.

This implies $\|T\| \geq b-a$

Combining ① & ② gives $\|T\| = b-a$.

□

3. Proof. For any $z \in (X/Y)^*$, $x \in X$.

$$(\pi^* z)(x) = z(\pi x).$$

Therefore, $|(\pi^* z)(x)| \leq \|z\|_{(X/Y)^*} \|\pi x\|_{X/Y} \leq \|z\|_{(X/Y)^*} \|x\|_X$

$$\Rightarrow \|\pi^* z\|_{X^*} \leq \|z\|_{(X/Y)^*}$$

It follows from the definition of the norms of $(X/Y)^*$, (X/Y) that $\forall n \in \mathbb{N}^*$, $\exists y_n \in X/Y$. s.t. $\|y_n\|_{X/Y} = 1$,

and $|z(y_n)| \in [\|z\|_{(X/Y)^*} - \frac{1}{n}, \|z\|_{(X/Y)^*}]$

and $\exists x_n \in X$. s.t. $\pi x_n = y_n$ and $\|x_n\|_X \in [1, 1+\frac{1}{n}]$,

Therefore, $\|\pi^* z\|_{X^*} \geq \dfrac{|z(\pi x_n)|}{\|x_n\|_X} \geq \dfrac{\|z\|_{(X/Y)^*} - \frac{1}{n}}{1+\frac{1}{n}}$.

Taking $n \to \infty$ gives $\|\pi^* z\|_{X^*} \geq \|z\|_{(X/Y)^*}$

So π^* is an isometry.

$\square$

4. Proof (i) It follows from the Hahn-Banach theorem that, for any $x \in X$, there exists $y_x \in B_{X^*}$. $\|y_x\|_{X^*} = 1$, $y_x(x) = \|x\|$. Since

$B_{X^*} \subseteq \bigcup_{k=1}^{n} B(x_k^*, r)$, then $\exists k_x \in \{1, \dots, n\}$, s.t. $\|y_x - x_{k_x}^*\|_{X^*} < r$.

$x_{k_x}^*(x) = y_x(x) + x_{k_x}^*(x) - y_x(x) \geq \|x\| - r\|x\|$.

Therefore, $\|Tx\|_\infty \geq (1-r)\|x\|$.

(ii) Since X is of finite dimension, so is X^*. Therefore $\forall 0<r<1, \exists \{x_k^*\}_{k=1}^n \subset B_{X^*}$ s.t. $B_{X^*} \subseteq \bigcup_{k=1}^n B(x_k^*, r)$. Then define $T: X \to \ell_\infty^n$ by $T(x) = (x_1^*(x), \cdots, x_n^*(x))$. Obviously, it is linear and $\|Tx\|_\infty \leq \sup_{1\leq k\leq n} \|x_k^*\|_{X^*} \cdot \|x\| \leq \|x\|$, i.e. $\|T\| \leq 1$.

(i) has proved that $\|x\| \leq \frac{1}{1-r} \|x\|$. Therefore, T is injective and $\|T\| \|T^{-1}\| < \frac{1}{1-r}$. Then choosing r small enough finishes the proof. $\square$