SUPPLEMENTARY NOTES ON TUTORIAL 1 (16/1/2019)

Let $\varphi : I \to \mathbb{R}$ be differentiable at some $c \in I$. Suppose in addition that $\varphi(c) = 0$. Prove that if $|\varphi|$ is differentiable at c, then $\varphi'(c) = 0$.

Proof. We prove by contraposition. Suppose that $\varphi'(c) \neq 0$. By definition of differentiability of φ at c, we have

$$\lim_{x \longrightarrow c} \frac{\varphi(x) - \varphi(c)}{x - c} = \varphi'(c) \neq 0.$$

Hence, by continuity of absolute value,

$$\lim_{x \longrightarrow c^+} \frac{|\varphi(x)| - |\varphi(c)|}{x - c} = \lim_{x \longrightarrow c^+} |\frac{\varphi(x)}{x - c}| = |\lim_{x \longrightarrow c^+} \frac{\varphi(x) - \varphi(c)}{x - c}| = |\varphi'(c)| > 0.$$

While

$$\lim_{x \longrightarrow c^-} \frac{|\varphi(x)| - |\varphi(c)|}{x - c} = -\lim_{x \longrightarrow c^-} |\frac{\varphi(x)}{x - c}| = -|\lim_{x \longrightarrow c^-} \frac{\varphi(x) - \varphi(c)}{x - c}| = -|\varphi'(c)| < 0.$$

Hence $\lim_{x \longrightarrow c} \frac{|\varphi(x)| - |\varphi(c)|}{x - c}$ does not exist and $|\varphi|$ is not differentiable at c.

Remark. In fact, this is a more unifying approach to the problem than the one I have presented during the tutorial.