Suggested solution of HW7

P.270 Q7: (a) $\sum a_n$ is absolutely convergent and b_n is bounded. Then there exists $M > 0$ such that

$$|b_n| \leq M \quad \text{for all} \quad n \in \mathbb{N}.$$

And there exists $C > 0$ such that for all $p \in \mathbb{N}$, we have

$$\sum_{n=1}^{p} |a_n| \leq C.$$

Thus, for any $p \in \mathbb{N}$,

$$\sum_{n=1}^{p} |a_n||b_n| \leq M \sum_{n=1}^{p} |a_n| \leq CM.$$

By monotone convergence theorem, $\sum a_n b_n$ is absolutely convergent.

(b) Take $a_n = \frac{(-1)^n}{n}$, $b_n = (-1)^n$,
Then $\sum a_n$ is conditional convergent, b_n is bounded, but $\sum a_n b_n$ is harmonic series which is divergent.

P.270 Q9:

If $\{a_n\}$ is a decreasing sequence of strictly positive numbers and if $\sum a_n$ is convergent, show that $na_n \to 0$ as $n \to \infty$.
Let $\epsilon > 0$ be given, by cauchy criterion, there exists $N \in \mathbb{N}$ such that for all $m \geq n > N$,

$$|\sum_{k=n}^{m} a_k| < \frac{\epsilon}{2}.$$

Take $n = N$, by the assumption, for any $m \geq N$

$$0 < (m - N + 1)\, a_m \leq \sum_{k=N}^{m} a_k < \frac{\epsilon}{2}$$

which implies $0 < ma_m < \epsilon/2 + (N-1)a_m$. Since $a_n \to 0$ as n goes to infinity. We can find $N' = N'(N, \epsilon)$ such that for all $n > N'$,

$$0 < a_n < \frac{\epsilon}{2(N-1)}.$$

Thus, for all $m > N' + N = \overline{N}$,

$$0 < ma_m < \epsilon/2 + (N-1)a_m < \epsilon.$$

P.270 Q11:

If $\{a_n\}$ is a sequence such that $\lim_n n^2 a_n$ exists. Let $l = \lim_n n^2 a_n$.
If $l \neq 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\frac{|l|}{2} \quad \leq \quad |n^2 a_n| \quad \leq \quad 2|l|\ .$$

Thus, $|a_n| \leq \frac{2|l|}{n^2}$ for all $n > N$.

If $l = 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|a_n| \leq \frac{1}{n^2}.$$

By comparison test, $\sum a_n$ converges absolutely.

P.276 Q4a:

Noted that $e > 2.7$, we have

$$2^n e^{-n} < \left(\frac{2}{2.7}\right)^n.$$

Since $\sum_{n=1}^{\infty} \left(\frac{2}{2.7}\right)^n$ converges. By comparison test, $\sum 2^n e^{-n}$ converges.

P.276 Q4c:

Let $x_n = e^{-\ln n} = \frac{1}{n}$,

$$\sum \frac{1}{n} \to +\infty.$$

Thus, the series diverges.

P.281 Q14:

It is given that the partial sum $s_n = \sum_{k=1}^{n} a_k$ satisfy $|s_n| \leq Mn^r$ for some $r < 1$. By Abel's Lemma, if $m > n$, then

$$\sum_{k=n+1}^{m} \frac{a_k}{k} = \left(\frac{s_m}{m} - \frac{s_n}{n+1}\right) + \sum_{k=n+1}^{m-1} \frac{s_k}{k(k+1)}.$$

Since $|s_n| \leq Mn^r$ for all n,

$$\left|\frac{s_m}{m}\right| \leq \frac{M}{m^{1-r}} \to 0 \quad \text{as} \quad m \to \infty.$$

Thus, for any $\epsilon > 0$, we can find a $N \in \mathbb{N}$ such that for all $m, n > N$

$$\left(\frac{s_m}{m} - \frac{s_n}{n+1}\right) < \epsilon.$$

Besides,

$$\sum_{k=n+1}^{m-1} \left|\frac{s_k}{k(k+1)}\right| \leq M \sum_{k=n+1}^{m-1} \frac{k^r}{k(k+1)} \leq M \sum_{k=n+1}^{m-1} \frac{1}{k^{2-r}}$$

where $2 - r > 1$. Since $\sum \frac{1}{n^{2-r}}$ converges, for all $\epsilon > 0$, there exists N' such that for all $m, n > N$,

$$M \sum_{k=n+1}^{m-1} \frac{1}{k^{2-r}} < \epsilon$$

Thus, by cauchy criterion, the series $\sum \frac{a_n}{n}$ converges.