Suggested solution of HW6

P.252 Q6:

$f_n(x) = \dfrac{1}{(1+x)^n}$ for $x \in [0,1]$.

If $x = 0$, then $f_n(0) = 1$ for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty} f_n(0) \to 1$.

If $x \in (0,1]$, $\frac{1}{1+x} < 1$. Thus $f_n(x) \to 0$ as $n \to \infty$. So we obtain the pointwise limit function $f : [0,1] \to \mathbb{R}$ at which

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \in (0,1]. \end{cases}$$

However, the convergence is non-uniform. We choose a sequence $\{x_n\} \subset [0,1]$ by $x_n = \dfrac{1}{n}$.

$$f_n(x_n) = \left(1 + \frac{1}{n}\right)^{-n} \to \frac{1}{e} > 0 \quad \text{as } n \to \infty.$$

P.253 Q9: $f_n(x) = \dfrac{x^n}{n}$ for $x \in [0,1]$. It is clear that f_n converges to $f = 0$ on $[0,1]$ uniformly. As for all $x \in [0,1]$,

$$0 \le f_n(x) = \frac{x^n}{n} \le \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

On the other hand, $f_n'(x) = x^{n-1}$ for $x \in [0,1]$. For $x \in [0,1)$, $\lim_{n\to\infty} f_n'(x) = 0$ and $f_n'(1) = 1$ $,\forall n \in \mathbb{N}$. Thus $g(1) = 1$ but $f'(1) = 0$.

P.253 Q12:

$$0 \le \int_1^2 \exp(-nx^2) \le \int_1^2 \exp(-n) = e^{-n}$$

Thus,

$$\lim_{n\to\infty} \int_1^2 \exp(-nx^2) = 0$$

P.253 Q17: For all n,

$$f_n(x) = \begin{cases} 1 & \text{if } x \in (0, 1/n) \\ 0 & \text{if } x \in [1/n, 1] \cup \{0\}. \end{cases}$$

Clearly, f_n is a discontinuous function. Let $c \in [0,1]$, $n \in \mathbb{N}$,

$$\begin{cases} f_{n+1}(c) = 1 = f_n(c) & \text{if } c \in \left(0, \frac{1}{n+1}\right) \\ f_{n+1}(c) = 0 \le 1 = f_n(c) & \text{if } c \in \left[\frac{1}{n+1}, \frac{1}{n}\right) \\ f_{n+1}(c) = 0 = f_n(c) & \text{if } c \in \left[\frac{1}{n}, 1\right] \cup \{0\}. \end{cases}$$

Thus, it is a decreaseing sequence. It can also be seen from above that $f_n(c) \to 0$ as n goes to infinity for any $c \in [0, 1]$. However, the convergence is not uniform. To see this, we can consider a sequence $\{x_n\} \subset [0, 1]$ at which $x_n = \frac{1}{2n}$.

$$f_n(x_n) = 1 \quad , \forall n \in \mathbb{N}.$$