Suggested solution of HW2

P.180 Q19:

Let $\epsilon > 0$ be given, there exists $\delta > 0$ such that if $0 < |x - y| < \delta$ and $x, y \in I = [a, b]$, then

$$\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \epsilon/2.$$

Interchange x and y, we obtain

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon/2.$$

So if $0 < |x - y| < \delta$ and $x, y \in I = [a, b]$,

$$|f'(x) - f'(y)| \leq \left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| + \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon.$$

So, f' is uniformly continuous.

P.187 Q3:

Let $\epsilon > 0$, there exists $\delta = \sqrt{\epsilon} > 0$ such that if $|x| < \delta$, then

$$|g(x)| = x^2 < \delta^2 = \epsilon.$$

Since

$$|f(x)| = x^2 \left| \sin \frac{1}{x} \right| \leq x^2 \ , \forall\, x \in \mathbb{R}.$$

By squeeze theorem and above result, $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x)$. Now we show that $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \sin \frac{1}{x}$ does not exist. Choose two sequences $\{a_n\}$ and $\{b_n\}$ at which

$$a_n = \frac{1}{2\pi n} \quad \text{and} \quad b_n = \frac{1}{2\pi n + \pi/2}.$$

Both sequences converge to 0 but

$$\frac{f(a_n)}{g(a_n)} = \sin 2\pi n = 0 \quad \text{and} \quad \frac{f(b_n)}{g(b_n)} = \sin(2\pi n + \pi/2) = 1.$$

It contradicts with the sequence criterion. So the limit doesn't exist.

P.196 Q4:

Let $f : [0, +\infty) \to \mathbb{R}$ by $f(x) = \sqrt{1 + x}$. Noted that f is twice differentiable on $(0, +\infty)$. By Taylor's theorem, for all $x > 0$ there exists $c \in (0, x)$ such that

$$f(x) = f(0) + f'(0)x + \frac{f''(c)}{2} x^2.$$

By usual computation, $f(0) = 1$, $f'(0) = \dfrac{1}{2}$ and $f''(c) = -\dfrac{1}{4(1+c)^{3/2}}$ which implies

$$1 + \frac{1}{2}\,x - \frac{1}{8}x^2 \leq f(x) \leq 1 + \frac{1}{2}\,x.$$