Suggested Solutions to Test

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1. (5 points) Let X be the space of all continous real valued functions defined on $[a, b]$. Suppose the X is endowed with the sup-norm, that is $\|f\| := \sup\{|f(x)| : x \in [a, b]\}$. Define $T : X \to \mathbb{R}$ by $T(f) = \int_a^b f(x)dx$ for $f \in X$. Show that $T \in X^*$ and find $\|T\|$.

Proof. Since for any $\alpha, \beta \in \mathbb{R}$ and $f, g \in X$

$$T(\alpha f + \beta g) = \int_0^t \alpha f(x) + \beta g(x)dx = \alpha \int_0^t f(x)dx + \beta \int_0^t g(x)dx = \alpha Tf + \beta Tg,$$

T is linear. Moreover, note that

$$|Tf| = |\int_a^b f(x)dx| \leq \|f\|(b-a),$$

Then T is bounded and $\|T\| \leq b - a$.
Taking $f \equiv 1$, then

$$|Tf| = |\int_a^b 1dx| = b - a,$$

which implies $\|T\| \geq b - a$.
Therefore, $T \in X^*$ and $\|T\| = b - a$.

2.Let X be a normed space. Supporse that there is a countable set $D := \{x_n : \|x_n\| = 1; n = 1, 2, \cdots\}$ dense in the closed unit sphere of X.
For each f and g in $B_{X^*} := \{f \in X^* : \|f\| \leq 1\}$, define

$$d(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} |f(x_n) - g(x_n)|.$$

(a) (5 points) Show that d is a metric on B_{X^*}.

(b) (10 points) Let $f \in B_{X^*}$. Show that for any $\varepsilon > 0$, we can find some elements $x_1, \cdots, x_N$ in D and $\delta > 0$ such that

$$d(f, g) < \varepsilon$$

whenever $g \in B_{X^*}$ with $|f(x_i) - g(x_i)| < \delta$ for all $i = 1, \cdots, N$.

Proof. Note that for any $f, g \in B_{X^*}$,

$$d(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} |f(x_n) - g(x_n)| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} (\|f\| + \|g\|)\|x_n\| \leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} < +\infty.$$

So. d is well defined.

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(a) It is easy to certify that d satisfy that $d(f,g) \geq 0$, $d(f,g) = d(g,f)$ and $d(f,g) \leq d(f,h) + d(h,g)$.
Now we claim that $d(f,g) = 0$ if and only if $f = g$.
Indeed, if $d(f,g) = 0$, then $|f(x_n) - g(x_n)| = 0$ i.e. $f(x_n) = g(x_n)$. Since D is dense in the closed unit sphere S of X, then for any $x \in S$, there exist a sequence $\{x_{n_k}\}$ in D such that $\lim_{k\to\infty} x_{n_k} = x$. By the continuity of f and g, one has

$$f(x) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} g(x_{n_k}) = g(x).$$

Finally, for any $x \in X - \{0\}$, $\frac{x}{\|x\|} \in S$, then it follows from the linearity of f and g that

$$f(x) = \|x\| f(\frac{x}{\|x\|}) = \|x\| g(\frac{x}{\|x\|}) = g(x).$$

It is clear that $f(0) = g(0)$. Therefore, $f = g$. On the other hand, it is obvious that $d(f,g) = 0$ when $f = g$.

(b) Given any $\varepsilon > 0$, choosing $N = 1 + [\log_2 \varepsilon]$, then

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n}|f(x_n) - g(x_n)| \leq \sum_{n=N+1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2^{N+1}} < \frac{\varepsilon}{2}$$

Let $\delta = \frac{\varepsilon}{2}$, then

$$\sum_{n=1}^{N-1} \frac{1}{2^n}|f(x_n) - g(x_n)| \leq \sum_{n=1}^{N-1} \frac{1}{2^n}\delta < \delta = \frac{\varepsilon}{2}$$

Therefore, $d(f,g) < \varepsilon$.

3. Let M be a closed subspace of a normed space X. Let $Q : X \to X/M$ be the quotient map. For each $x \in X$, the distance between x and M is defined by $d(x,M) := \inf\{\|x - m\| : m \in M\}$.

(a) (5 points) Show that if $\bar{F} \in (X/M)^*$, then $\|\bar{F}\| = \|\bar{F} \circ Q\|$.

(b) (10 points) If $a \notin M$, show that there is $f \in X^*$ such that $f(M) \equiv 0; f(a) = 1$ and $\|f\| = \frac{1}{d(a,M)}$.

Proof.

(a) Let $\bar{F} \in (X/M)^*$. For any $x \in X$, set $\bar{x} = Q(x)$. Then, by the definition of norm on quotient space X/M

$$\|\bar{x}\|_{X/M} = \inf_{m\in M} \|x - m\| \leq \|x - 0\| = \|x\|.$$

Thus, $\|\bar{F} \circ Q(x)\| = \|\bar{F}(\bar{x})\| \leq \|\bar{F}\|\|\bar{x}\|_{X/M} \leq \|\bar{F}\|\|x\|$.
Therefore, $\bar{F} \circ Q$ is bounded and $\|\bar{F} \circ Q\| \leq \|\bar{F}\|$.
On the other hand, for any $\bar{x} \in X/M$, there exists a $m_0 \in M$ such that $\|x - m_0\| \leq \|\bar{x}\|_{X/M} + \varepsilon$. Then,

$$\|\bar{F}(\bar{x})\| = \|\bar{F}(Qx)\| = \|\bar{F}(Q(x-m_0))\| = \|\bar{F}\circ Q(x-m_0)\| \leq \|\bar{F}\circ Q\|\|x-m_0\| \leq \|\bar{F}\circ Q\|(\|\bar{x}\|_{X/M}+\varepsilon).$$

Since ε is arbitary, $\|\bar{F}\| \leq \|\bar{F} \circ Q\|$.

(b) Let $a \notin M$. Then $\|\bar{a}\|_{X/M} = d(a, M) > 0$. Set $X_0 = \{\alpha \bar{a}\}$ and define $\bar{F}_0$ on X_0 by $\bar{F}_0(\alpha \bar{a}) = \alpha$. Then $\bar{F}_0$ is linear and $\bar{F}_0(\bar{a}) = 1$. Since

$$|\bar{F}_0(\alpha \bar{a})| = |\alpha| \leq \|\alpha \bar{a}\| \frac{1}{\|\bar{a}\|} = \frac{1}{d(a, M)} \|\alpha \bar{a}\|.$$

F_0 is bounded. Then

$$|\alpha| = |\bar{F}_0(\alpha \bar{a})| \leq \|F_0\| \|\alpha \bar{a}\| \leq \|F_0\| \|\bar{a}\| |\alpha|$$

which implies that $\|F_0\| \geq \dfrac{1}{d(a, M)}$. So, F_0 is a linear bounded functional on X_0. By Hahn-Banach Theorem, there exists a bounded linear functional $\bar{F}$ on X/M such that

$$\bar{F}(\bar{a}) = \bar{F}_0(\bar{a}) = 1, \quad \text{and} \quad \|\bar{F}\| = \|\bar{F}_0\| = \frac{1}{d(a, M)}.$$

Set $f = \bar{F} \circ Q$, then $f(a) = \bar{F} \circ Q(a) = \bar{F}(\bar{a}) = 1$, $f(M) = \bar{F}(Q(M)) = \bar{F}(0) = 0$. Moreover, by (a), $\|f\| = \|\bar{F}\| = \dfrac{1}{d(a, M)}$.