Suggested Solution to Homework 5

Yu Mei†

P238, 9.(Annihilator) Let X and Y be normed spaces, $T : X \to Y$ a bounded linear operator and $M = \overline{\mathscr{R}(T)}$, the closure of the range of T. Show that

$$M^a = \mathscr{N}(T^\times).$$

Proof. On the one hand, let $f \in M^a \subset Y'$, then

$$(T^\times f)(x) = f(Tx) = 0, \quad x \in X \text{ such that } Tx \in \mathscr{R}(T) \subseteq M.$$

So, $f \in \mathscr{N}(T^\times)$ which yields that $M^a \subseteq \mathscr{N}(T^\times)$. On the other hand, let $g \in \mathscr{N}(T^\times)$, then, for any $y \in M$, there exists a sequence of $\{x_n\} \in X$ such that $y = \lim_{n\to+\infty} Tx_n$. Since $g \in \mathscr{N}(T^\times)$ is continuous, we have

$$g(y) = g(\lim_{n\to+\infty} Tx_n) = \lim_{n\to+\infty} g(Tx_n) = \lim_{n\to+\infty} (T^\times g)(x_n) = 0.$$

So, $g \in M^a$ which yields that $\mathscr{N}(T^\times) \subseteq M^a$.
Therefore, $M^a = \mathscr{N}(T^\times)$. □

P239, 10. Let B be a subset of the dual space X' of a normed space X. The annihilator ${}^aB$ of B is defined to be

$${}^aB = \{x \in X | f(x) = 0 \text{ for all } f \in B\}.$$

Show that, in the above problem,

$$\mathscr{R}(T) \subset {}^a\mathscr{N}(T^\times).$$

What does this mean with respect to the task of solving an equation $Tx = y$?

Proof. Let $y = Tx \in \mathscr{R}(T)$. Then, for any $f \in \mathscr{N}(T^\times)$, since $T^\times f = 0$, we have

$$f(y) = f(Tx) = (T^\times f)(x) = 0.$$

which yields that $y \in {}^a\mathscr{N}(T^\times)$. So, $\mathscr{R}(T) \subset {}^a\mathscr{N}(T^\times)$.

This means that a necessary condition for the existence of solution to $Tx = y$ is that $f(y) = 0, \forall f \in \mathscr{N}(T^\times)$. □

P246, 8. Let M be any subset of a normed space X. Show that an $x_0 \in X$ is an element of $A = \overline{\text{span}M}$ if and only $f(x_0) = 0$ for every $f \in X'$ such that $f\big|_M = 0$.

Proof. Assume $x_0 \in \overline{\text{span}M}$. Let $f \in X'$ and $f\big|_M = 0$. Then, by linearity, $f(x) = 0$, for any $x \in \text{span}M$. Morevoer, since f is bounded, so is continous. Therefore, $f(x_0) = \lim_{n\to+\infty} f(x_n) = 0$, where (x_n) is a sequence in span$M$ converging to x_0.

On the other hand, assume $f(x_0) = 0$ for every $f \in X'$ such that $f\big|_M = 0$. We claim that $x_0 \in \overline{\text{span}M}$. Otherwise, suppose $x_0 \notin Z := \overline{\text{span}M}$, then $\text{dist}(x_0, Z) = \delta > 0$. Then by Lemma 4.6-7, $\exists\ \tilde{f}$ in X s.t $\|\tilde{f}\| = 1$, $\tilde{f}(x_0) = \delta$ and $\tilde{f}\big|_Z = 0$. Thus $\tilde{f}\big|_M = 0$. A contradiction, since $\tilde{f}(x_0) \neq 0$. □

† Email address: ymei@math.cuhk.edu.hk. (Any questions are welcome!)