Suggested Solution to Homework 3

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P71, 8. If in a normed space X, absolute convergence of any series always imolies convergence of that series, show that X is complete.

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. To prove that X is complete, it suffices to show there exists a subsequence $\{x_{n_k}\}$ of the Cauchy sequence $\{x_n\}$ which converges. (Refer to P32, Q2. in HW1.)

Since $\{x_n\}$ is a Cauchy sequence, then for $\epsilon_k = \dfrac{1}{2^k}, \forall k \in \mathbb{N}$, there exists N_k such that

$$\|x_n - x_m\| < \epsilon_k, \forall n, m > N_k.$$

Thus, $\exists n_{k+1} > n_k > N_k$ s.t. $\|x_{n_{k+1}} - x_{n_k}\| < \dfrac{1}{2^k}$.

Set $y_k = x_{n_k}$, then $\sum_{i=1}^{\infty} \|y_{i+1} - y_i\| = \sum_{i=1}^{\infty} \|x_{n_{i+1}} - x_{n_i}\| < \sum_{i=1}^{\infty} \dfrac{1}{2^i} = 1$. So $\{y_{i_1} - y_i\}$ is absolutely convergent which implies convergence of the series $\sum_{i=1}^{\infty}(y_{i+1} - y_i)$.

Therefore, $y_k = y_1 + \sum_{i=1}^{k}(y_{i+1} - y_i)$. converge,i.e. $\{x_{n_k}\}$ converge. □

P71, 9. Show that in a Banach space, and absolutely convergent series is convergent.

Proof. Let X be a Banach space. Then for any $\{x_n\} \subset X$, $\| \sum_{k=n}^{m+n} x_k\| \leq \sum_{k=n}^{m+n} \|x_k\| \to 0$ as $n \to +\infty$. That is $\|s_{n+m} - s_n\| \to 0$ as $n \to +\infty$, where $s_n = \sum_{k=1}^{n} x_k$. Thus, s_n is a Cauchy sequence in X. By the completeness of X, s_n converges. So, $\sum_{k=1}^{\infty} x_k < +\infty$. □

P76, 8. Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in Prob. 8, Sec. 2.2, satisfy

$$\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1.$$

Proof. Since

$$\|x\|_1^2 = (\sum_{i=1}^{n} |\xi_i|)^2 \geq \sum_{i=1}^{n} |\xi_i|^2 = \|x\|_2^2,$$

then $\|x\|_1 \geq \|x\|_2$.

On the other hand,

$$\frac{1}{\sqrt{n}}\|x\|_1 = \sum_{i=1}^{n} \frac{1}{\sqrt{n}}|\xi_i| \leq (\sum_{i=1}^{2} |\xi_i|^2)^{\frac{1}{2}} (\sum_{i=1}^{n} (\frac{1}{\sqrt{n}})^2)^{\frac{1}{2}} = \|x\|_2.$$

Therefore,

$$\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1.$$

□

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