MATH 4010 | 04/02/2016

Tutorial 3

- Examples of Banach spaces

Eg 1. Euclidean space $\mathbb{R}^n$ with stantord norm

$$x=(x_1,x_2,\cdots,x_n) \longmapsto \|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$$

$(\mathbb{R}^n, \|\cdot\|)$ is a Banach space.

Eg 2. $L^p(\Omega,\mathcal{F},\mu)$ is a Banach space, where $1\leq p\leq\infty$.

- For $1\leq p<\infty$, $L^p(\Omega,\mathcal{F},\mu)=\{f:\Omega\to\mathbb{R} \mid f$ is measurable and $\int_\Omega |f(x)|^p d\mu(x)<+\infty\}$

$$\text{with } \|f\|_{L^p} (=\|f\|_p) = \left(\int_\Omega |f(x)|^p d\mu\right)^{\frac{1}{p}}$$

- For $p=\infty$, $L^\infty(\Omega,\mathcal{F},\mu)=\{f:\Omega\to\mathbb{R} \mid f$ is measuable and $|f(x)|\leq M$ a.e. on Ω for some constant $0<M<\infty\}$

$$\text{with } \|f\|_{L^\infty} (=\|f\|_\infty) = \inf\{M : |f(x)|\leq M \text{ a.e. on } \Omega\}$$

Here $(\Omega,\mathcal{F},\mu)$ denotes a measure space, i.e. Ω is a set and

(i) $\mathcal{F}$ is a σ-algebra in Ω, i.e. $\mathcal{F}$ is a collection of subsets of Ω s.t.

(a) $\phi\in\mathcal{F}$;

(b) $A\in\mathcal{F} \Rightarrow A^c\in\mathcal{F}$

(c) $\bigcup_{n=1}^{\infty} A_n\in\mathcal{F}$ whenever $A_n\in\mathcal{F}, \forall n$.

(ii) μ is a measure, i.e. $\mu:\mathcal{F}\longmapsto[0,\infty]$ satisfies

(a) $\mu(\phi)=0$

(b) $\mu(\bigcup_{n=1}^{\infty} A_n)=\sum_{n=1}^{\infty}\mu(A_n)$ whenever $\{A_n\}$ is a disjoint countable subsets of $\mathcal{F}$.

(iii) Ω is σ-finite, i.e. $\exists$ countable subsets $\{\Omega_n\}$ of $\mathcal{F}$ s.t.

$$\Omega=\bigcup_{n=1}^{\infty}\Omega_n \text{ and } \mu(\Omega_n)<+\infty, \forall n.$$

<u>Remarks</u>: 1° Recall that for $f,g\in L^p(\Omega,\mu)$, $f\sim g$ if $f=g$ a.e. on Ω.

2°. For $f\in L^\infty(\Omega,\mu)$, $|f(x)|\leq\|f\|_{L^\infty}$ a.e. on Ω.

Indeed, by the def. of inf., $\forall n\in\mathbb{N}$,

$|f(x)|\leq\|f\|_{L^\infty}+\frac{1}{n}$ a.e. $\Leftrightarrow$ $\mu(E_n)=0$ with $E_n=\{x\in\Omega: |f(x)|>\|f\|_{L^\infty}+\frac{1}{n}\}$

Note that $E:=\{x\in\Omega: |f(x)|>\|f\|_{L^\infty}\}=\bigcup_{n=1}^{\infty}E_n$

Then, $\mu(E)\leq\sum_{n=1}^{\infty}\mu(E_n)=0$.

3°, $\Omega=\mathbb{R}^n$, μ - Lebesgue measure $\Rightarrow L^p(\mathbb{R}^n)$

$\Omega=\mathbb{Z}^+$, μ - counting measure $\Rightarrow \ell^p$.

Pf: Step 1: $L^p(\Omega, \mathcal{F}, \mu)$ is a normed space.

It is clear that $\begin{cases} (i) \ \|f\|_{L^p} \geq 0 \text{ and } \|f\|_{L^p} = 0 \text{ iff } f = 0 \text{ a.e.} \\ (ii) \ \|\alpha f\|_{L^p} = |\alpha| \, \|f\|_{L^p} \end{cases}$ (Note that Equivalent class is introduced as shown in Remark 1°)

It sufficies to show the triangle inequality holds true, i.e.

(M) $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$

This is the well-known Minkowski inequality.

This is obvious for $p=1$ or ∞. W.L.O.G we assume $1<p<\infty$.

To prove it, we first prove the Hölder inequality, i.e.

(H) $\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$, $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p, q < \infty$ ($1 \leq p, q \leq \infty$).

(H) is a consequence of Young inequality

(Y) $ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$ for $a, b \geq 0$.

Indeed, by applying Young inequality,

$$\|fg\|_{L^1} = \int_\Omega |f||g| \, d\mu \leq \int_\Omega \frac{1}{p}|f|^p + \frac{1}{q}|g|^q \, d\mu = \frac{1}{p}\|f\|_{L^p}^p + \frac{1}{q}\|g\|_{L^q}^q.$$

Thus, if $\|f\|_{L^p} = \|g\|_{L^q} = 1$, then $\|fg\|_{L^1} \leq \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_{L^p}\|g\|_{L^q}$

Otherwise, replacing f by $\frac{f}{\|f\|_{L^p}}$ and g by $\frac{g}{\|g\|_{L^q}}$

we have, $\left\|\frac{f}{\|f\|_{L^p}}\right\|_{L^p} = 1$, $\left\|\frac{g}{\|g\|_{L^q}}\right\|_{L^q} = 1$ and $\left\|\frac{f}{\|f\|_{L^p}} \cdot \frac{g}{\|g\|_{L^q}}\right\| \leq \frac{1}{p} + \frac{1}{q} = 1$

The Young inequality follows from the concavity of log fcn, i.e.

$$\log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \geq \frac{1}{p}\log a^p + \frac{1}{q}\log b^q \geq \log a + \log b = \log ab.$$

Now, we prove (M),

$\forall f, g \in L^p$,

$$\|f+g\|_{L^p}^p = \int_\Omega |f+g|^p \, d\mu \leq \int_\Omega |f+g|^{p-1}|f| \, d\mu + \int_\Omega |f+g|^{p-1}|g| \, d\mu$$

$$\leq \left[\int_\Omega (|f+g|^{p-1})^q\right]^{\frac{1}{q}} \|f\|_{L^p} + \left[\int_\Omega (|f+g|^{p-1})^q\right]^{\frac{1}{q}} \|g\|_{L^p}$$

$$= \|f+g\|_{L^p}^{p-1} \|f\|_{L^p} + \|f+g\|_{L^p}^{p-1} \|g\|_{L^p}$$

Since $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \Rightarrow q = \frac{p}{p-1}$

So, $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$

Step 2. $L^p(\Omega, \mathcal{F}, \mu)$ is complete.

Note that the induced metric, $d(f,g) := \|f-g\|_{L^p}$.

<u>Case 1: $1 \le p < \infty$.</u>

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^p(\Omega, \mathcal{F}, \mu)$.

It sufficies to show that there exist a subsequence $\{f_{n_k}\}$ converges in L^p.

We extract a subsequence $\{f_{n_k}\}$ satisfiing

$$\|f_{n_{k+1}} - f_{n_k}\|_{L^p} \le \frac{1}{2^k}, \quad \forall k \ge 1.$$

Now, we prove f_{n_k} converges in L^p.

Define
$$f(x) = f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k})$$
$$g(x) = |f_{n_1}| + \sum_{k=1}^{\infty}|f_{n_{k+1}} - f_{n_k}|$$

Then, the partial sum of f, g are
$$S_K f = f_{n_1} + \sum_{k=1}^{K-1}(f_{n_{k+1}} - f_{n_k}) = f_{n_K}$$
$$S_K g = |f_{n_1}(x)| + \sum_{k=1}^{K-1}|f_{n_{k+1}}(x) - f_{n_k}(x)|$$

Note that $\|S_K g\|_{L^p} \le \|f_{n_1}\|_{L^p} + \sum_{k=1}^{K-1}\|f_{n_{k+1}} - f_{n_k}(x)\|_{L^p} \le \|f_{n_1}\|_{L^p} + \sum_{k=1}^{K} 2^{-k}$

By the monotone convergence theorem (M.C.T.)
$$S_K g \to g \text{ a.e. on } \Omega \text{ and } \int_X g^p \, d\mu < +\infty.$$

So, f converges a.e. and $f \in L^p$ because of $|f(x)| \le g(x)$ a.e.

To prove $f_{n_K} \to f$ in L^p, as $K \to \infty$

$$|f_{n_K} - f|^p = |S_K f - f|^p \le 2^p(|S_K f|^p + |f|^p) \le 2^{p+1}|g|^p$$

and $f_{n_K} \to f$ a.e. as $K \to \infty$

By D.C.T., $\|f_{n_K} - f\|_{L^p} \to 0$ as $K \to \infty$.

<u>Case 2. $p = \infty$</u>

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^\infty(\Omega, \mathcal{F}, \mu)$.

Then, $\forall k \in \mathbb{N}$, $\exists N_k$ s.t. $\forall m, n \ge N_k$, $\|f_n - f_m\|_{L^\infty} < \frac{1}{k}$.

i.e. $|f_n(x) - f_m(x)| < \frac{1}{k}$ on $\Omega \setminus E_k$ with $\mu(E_k) = 0$.
$\forall n, m \ge N_k$

Set $E = \bigcup_{k=1}^{\infty} E_k$, then $\mu(E) = 0$ and $\forall\, x \in \Omega \setminus E$,

$$|f_n(x) - f_m(x)| < \frac{1}{k}$$

Thus. $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. So $f_n(x) \to f(x)$; $\forall\, x \in \Omega \setminus E$ for some $f(x)$.

Passing the limit as $m \to \infty$

$$|f_n(x) - f(x)| \leq \frac{1}{k}, \quad \forall\, x \in \Omega \setminus E, \quad \forall\, n \geq N_k$$

$$\Rightarrow \quad \|f_n(x) - f(x)\|_{L^\infty} \leq \frac{1}{k}, \ \forall\, x \in \Omega \setminus E, \ n \geq N_k, \ f \in L^\infty$$

Therefore $f_n(x) \to f(x)$ in L^∞.

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