Math 2050A: Analysis I: Test 06 Nov 2018. 8:30-10:00

Answer ALL Questions

1. (5 points) Using the ε-N argument show $\lim_{n\to\infty} \frac{n^2}{2n^2+n+1} = \frac{1}{2}$.

   Proof: We first note that for each positive integer n, we have

$$|\frac{1}{2} - \frac{n^2}{2n^2+n+1}| = \frac{n+1}{2(2n^2+n+1)} \leq \frac{n+1}{4n^2} \leq \frac{1}{n}.$$

   So, for any $\varepsilon > 0$ we take a positive integer N so that $\frac{1}{\varepsilon} < N$. Then we have

$$|\frac{1}{2} - \frac{n^2}{2n^2+n+1}| < \varepsilon$$

   for any $n \geq N$. □

2. (5 points) Let $x_n = \sum_{k=1}^{n} \frac{1}{3k+1}$ for $n = 1, 2, ...$ Using the definition of a Cauchy sequence show that (x_n) is not a Cauchy sequence.

   Proof: Notice that for $m > n$, we have

$$|x_m - x_n| = \sum_{k=n+1}^{m} \frac{1}{3k+1} > \frac{1}{3} \sum_{k=n+1}^{m} \frac{1}{k+1} > \frac{m-n}{3m}$$

   Thus, if we put $\varepsilon = \frac{1}{6}$, then for any positive integer N, we have $|x_{2N} - x_N| > \frac{1}{6} = \varepsilon$. Hence, (x_n) is not a Cauchy sequence. □

3. (10 points) Let (x_n) and (y_n) be convergent sequences.

   (i) If there are constants a and b such that $x_n < a < b < y_n$ for all n, show that

$$\lim x_n < \lim y_n.$$

   (by using the definition of limit).

   (ii) If we only assume that there is a constant c so that $x_n < c < y_n$ for all n, does the assertion in (i) still hold?

   Proof: (i): Put $L := \lim x_n$ and $M := \lim y_n$. We are going to prove by contradiction. We first notice that it is impossible for $L = M$ because we have $y_n - x_n > b - a > 0$ for all n. Now we suppose $M < L$. Thus, we can choose $\varepsilon > 0$ so that $M + \varepsilon < L - \varepsilon$. Then by the definition of limit, there are positive integers N_1 and N_2 such that $y_n < M + \varepsilon$ for all $n \geq N_1$ and $x_n > L - \varepsilon$ for all $n \geq N_2$. Now if we consider $N = \max(N_1, N_2)$, then we have

$$b < y_N < M + \varepsilon < L - \varepsilon < x_N < a.$$

It contradicts to the assumption of $a < b$. The proof is finished.
(ii): The answer is negative. For example, consider $x_n = \frac{-1}{n}$ and $y_n = \frac{1}{n}$. Then we have $x_n < 0 < y_n$ for all n but $\lim x_n = \lim y_n = 0$. □

4. (i) (5points) State the definition of a compact subset of $\mathbb{R}$.
   (ii) (10 points) Show that if A is a non-empty compact subset of $\mathbb{R}$, then there exists an element $z \in A$ such that $|z| = \max\{|x| : x \in A\}$.
   (iii) (5 points) In part *(ii)* if A is only assumed to be bounded, does the assertion still hold?

Proof: *(i)*: A set A is said to be compact if every sequence in A has a subsequence that converges to some element in A.
(ii): Let $S := \{|x| : x \in A\}$. Recall a fact that A is a compact subset of $\mathbb{R}$ if and only if it is a closed and bounded set. So, the set S is a non-empty bounded subset of $\mathbb{R}$. The axiom of completeness tells us that the supremum of S exists. Put $L := \sup S$. From this, we see that for each $n = 1, 2, ..$, there exists an element $x_n \in A$ satisfying $L - \frac{1}{n} < |x_n| < L + \frac{1}{n}$. This implies that $\lim |x_n| = L$.
On the other hand, then by using the compactness of A, we can find a subsequence (x_{n_k}) of (x_n) such that the limit $z := \lim_k x_{n_k}$ exists in A and thus, $|z| = \lim_k |x_{n_k}|$. Note that $\lim_k |x_{n_k}| = L$ because $\lim_n |x_n| = L$. Therefore, we have $|z| = L$ as required.
(iii) Consider $A = (0, 1)$. Then A is a bounded set. Notice that for each element $z \in (0, 1)$, we must have $z < \frac{1}{2}(z + 1) < 1$. So, the maximum of A does not exist. □

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