Suggested Solution to Homework 7

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P207, 11.(Unitary equivalence) Let S and T be linear operators on a Hilbert space H. The operator S is said to be unitarily equivalent to T if there is a unitary operator U on H such that

$$S = UTU^{-1} = UTU^{*}.$$

If T is self-adjoint, show that S is self-adjoint.

Proof. Suppose T is self-adjoint and S is unitarily equivalent to T. Then, it follows from the properties of Hilbert-adjoint operators that

$$S^{*} = (UTU^{*})^{*} = (U^{*})^{*}(UT)^{*} = UT^{*}U^{*} = UTU^{*} = S.$$

Therefore, S is also self-adjoint. □

P208, 13. If $T_n : H \to H (n = 1, 2, \cdots)$ are normal linear operators and $T_n \to T$, show that T is a normal linear operator.

Proof. It is clear that T is a bounded linear operator. It follows from the properties of Hilbert-adjoint operators that

$$\begin{aligned}
\|T_n^{*}T_n - T^{*}T\| &\le \|T_n^{*}T_n - T_n^{*}T\| + \|T_n^{*}T - T^{*}T\| \\
&\le \|T_n^{*}\|\|T_n - T\| + \|T_n^{*} - T^{*}\|\|T\| \\
&= \|T_n\|\|T_n - T\| + \|T_n - T\|\|T\| \to 0, \quad \text{as} \quad n \to +\infty,
\end{aligned}$$

since $T_n \to T$. Then, since T_n is normal, i.e. $T_nT_n^{*} = T_n^{*}T_n$, it holds that

$$\begin{aligned}
\|TT^{*} - T^{*}T\| &\le \|TT^{*} - T_nT_n^{*}\| + \|T_nT_n^{*} - T^{*}T\| \\
&= \|(T^{*}T - T_n^{*}T_n)^{*}\| + \|T_n^{*}T_n - T^{*}T\| \\
&= 2\|T_n^{*}T_n - T^{*}T\| \to 0, \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

Therefore, $TT^{*} = T^{*}T$, i.e. T is normal. □

P208, 14. If S and T are normal linear operators satisfying $ST^{*} = T^{*}S$ and $TS^{*} = S^{*}T$, show that their sum $S + T$ and product ST are normal.

Proof.

$$\begin{aligned}
(S + T)(S + T)^{*} &= (S + T)(S^{*} + T^{*}) \\
&= SS^{*} + ST^{*} + TS^{*} + TT^{*} \\
&= S^{*}S + T^{*}S + S^{*}T + T^{*}T \\
&= (S^{*} + T^{*})(S + T) \\
&= (S + T)^{*}(S + T),
\end{aligned}$$

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and

$$(ST)(ST)^* = STT^*S^* = ST^*TS^* = T^*SS^*T = T^*S^*ST = (ST)^*ST.$$

Therefore, $S+T, ST$ are normal. □

P208, 15. Show that a bounded linear operator $T : H \to H$ on a complex Hilbert space H is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$. Using this, show that for a normal linear operator,

$$\|T^2\| = \|T\|^2.$$

Proof. By the definition of adjoint operator,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle$$

and

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle x, TT^*x\rangle.$$

Then T is normal, i.e. $TT^* = T^*T$ if and only if $\|Tx\| = \|T^*x\|$.
Since, for any $x \in H$,

$$\|T^2x\| \le \|T\|\|Tx\| \le \|T\|\|T\|\|x\|,$$

it yields that

$$\|T^2\| \le \|T\|^2.$$

On the other hand, for any $x \in H$, it holds that

$$\begin{aligned}\|T^2x\|^2 &= \langle T^2x, T^2x\rangle = \langle Tx, T^*T^2x\rangle \\ &= \langle Tx, TT^*Tx\rangle = \langle T^*Tx, T^*Tx\rangle \\ &= \|T^*Tx\|^2.\end{aligned}$$

Then,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \le \|T^*Tx\|\|x\| \le \|T^2x\|\|x\| \le \|T^2\|\|x\|^2,$$

that is,

$$\|T\|^2 \le \|T^2\|.$$

Hence, $\|T^2\| = \|T\|^2$. □