Suggested Solution to Homework 6

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P175, 6. Let H be a separable Hilbert space and M a countable dense subset of H. Show that H contains a total orthonormal sequence which can be obtained from M by the Gram-Schmidt process.

Proof. W.L.O.G. assume H is infinite dimensional separable Hilbert space. $M = \{x_n\}_{n=1}^{\infty}$ is a countable dense subset of H. Then these exist a linear independent subsequence $N = x_{n_k}{}_{k=1}^{\infty}$ of M dense in H, otherwise, H is finite dimensional so that the conclusion is trivial. Using the Gram-Schmidt process, we can obtain an orthonormal sequence e_{n_k} by $e_{n_k} = \frac{v_{n_k}}{\|v_{n_k}\|}$ with $v_{n_k} = x_{n_k} - \sum_{j=1}^{k-1} \langle x_{n_k}, e_{n_j} \rangle e_{n_j}$. Moreover, $\overline{span e_{n_k}} = \overline{span x_{n_k}} = H$. Therefore, e_{n_k} is total orthonormal sequence obtained from M in H. □

P200, 4. Let H_1 and H_2 be Hilbert spaces and $T : H_1 \to H_2$ a bounded linear operator. If $M_1 \subset H_1$ and $M_2 \subset H_2$ are such that $T(M_1) \subset M_2$, show that $M_1^{\perp} \supset T^*(M_2^{\perp})$.

Proof. Let $z \in T^*(M_2^{\perp})$. Then, there exist $y \in M_2^{\perp}$ such that $z = T^*y$. By the definition of Hilbert-adjoint operator, for any $x \in M_1^{\perp}$, one has,

$$\langle x, T^*y \rangle = \langle Tx, y \rangle = 0,$$

since $Tx \in T(M_1) \subset M_2$ and $y \in M_2^{\perp}$. Therefore, $z = T^*y \in M_1^{\perp}$ so that $M_1^{\perp} \supset T^*(M_2^{\perp})$. □

P200, 5. Let M_1 and M_2 in Prob. 4 be closed subspaces. Show that then $T(M_1) \subset M_2$ if and only if $M_1^{\perp} \supset T^*(M_2^{\perp})$.

Proof. By the conclusion of Prob. 4, one has that $T(M_1) \subset M_2$ implies $M_1^{\perp} \supset T^*(M_2^{\perp})$.

Now assume $M_1^{\perp} \supset T^*(M_2^{\perp})$, where M_1 and M_2 are closed subspaces of Hilbert spaces H_1 and H_2 respectively, one need to show that $T(M_1) \subset M_2$. We use the argument by contradiction. Suppose that $T(M_1)$ is not a subset of M_2. Then there exist $0 \neq x \in T(M_1) - M_2$, since $0 \in T(M_1) \cap M_2$. Note that M_2 is a closed subspace of Hilbert space H_2, it yields that $x = y + z$ for some $y \in M_2$ and $0 \neq z \in M_2^{\perp}$. Moreover $x = Tw$ for some $w \in M_1$. Since $M_1^{\perp} \supset T^*(M_2^{\perp})$, $T^*z \in M_1^{\perp}$, it follows from the definition of Hilbert-adjoint operator that

$$0 = \langle w, T^*z \rangle = \langle Tw, z \rangle = \langle x, z \rangle = \langle y, z \rangle + \langle z, z \rangle = \langle z, z \rangle$$

Therefore $z = 0$, which is a contradiction. □

P200, 6. If $M_1 = \mathcal{N}(T) = \{x | Tx = 0\}$ in Prob. 4, show that

$$(a) \quad T^*(H_2) \subset M_1^{\perp}, \quad (b) \quad [T(H_1)]^{\perp} \subset \mathcal{N}(T^*), \quad (c) \quad M_1 = [T^*(H_2)]^{\perp}.$$

Proof.

(a) Note that M_1 is a closed subspace of Hilbert space H_1. Since $T(M_1) = \{0\}$ and $H_2 = \{0\}^{\perp}$, taking $M_2 = \{0\}$ in Prob. 4, one has $T^*(H_2) \subset M_1^{\perp}$.

(b) Let $x \in [T(H_1)]^{\perp}$. Then, $\langle y, x \rangle = 0$ for any $y = Tz \in T(H_1)$. It follows from the definition of adjoint operator that

$$0 = \langle Tz, x \rangle = \langle z, T^*x \rangle, \quad \text{for any} \quad z \in H_1.$$

Therefore, $T^*x = 0$, i.e. $x \in \mathcal{N}(T^*)$. Hence, (b) is valid.

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(c) Taking orthogonal complement in (a) yields that $M_1 \subset [T^*(H_2)]^\perp$, since $M_1 = \mathcal{N}(T)$ is a closed subspace. It suffice to show that $[T^*(H_2)]^\perp \subset M_1$. Indeed, let $x \in [T^*(H_2)]^\perp$. Then

$$0 = \langle x, T^* y\rangle = \langle Tx, y\rangle, \quad \text{for any} \quad y \in H_2,$$

which implies that $Tx = 0$,i.e. $x \in M_1 = \mathcal{N}(T)$.

□