Sequence of Functions

Definition (c.f. Definition 8.1.1)**.** Let (f_n) be a sequence of real-valued functions defined on $A \subseteq \mathbb{R}$. The sequence (f_n) is said to **converge (pointwisely)** to a function f on A if for each $x \in A$, $(f_n(x))$ converges to $f(x)$. In this case, we denote

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{or} \quad f = \lim_{n\to\infty} f_n.$$

Definition (c.f. Definition 8.1.4)**.** Let (f_n) be a sequence of real-valued functions defined on $A \subseteq \mathbb{R}$. The sequence (f_n) is said to **converge uniformly** to a function f on A if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ and $x \in A$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Remark. Notice that in ε-N notation, N depends on ε and x for pointwise convergence. For uniform convergence, N depends on ε **ONLY**. Also, uniform convergence implies pointwise convergence. (Prove it as an exercise!)

Example 1 (c.f. Section 8.1, Ex.8 & Ex.18)**.** Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by

$$f_n(x) = xe^{-nx}.$$

Show that (f_n) converges uniformly on $[0, \infty)$.

Solution. We first find the pointwise limit of f_n. If $x = 0$, then $f_n(x) = 0$ for all n. Hence

$$\lim_{n\to\infty} f_n(0) = 0.$$

If $x > 0$, note that $e^{-nx} \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} f_n(x) = x \cdot \lim_{n\to\infty} e^{-nx} = 0.$$

Now we show that (f_n) converges uniformly to 0. Firstly, we need to find the maximum of $f_n(x) = xe^{-nx}$ on $[0, \infty)$ for each n. Differentiate f_n gives

$$f_n'(x) = (1 - nx)e^{-nx}.$$

Hence it has only one critical point at $x = 1/n$. Now at the endpoints and critical point,

$$f_n(0) = 0, \quad f_n(1/n) = \frac{1}{ne} \quad \text{and} \quad \lim_{x\to\infty} f_n(x) = 0.$$

It follows that the maximum value of f_n is $1/ne$. Hence

$$|xe^{-nx} - 0| \leq \frac{1}{ne}, \quad \forall n, \quad \forall x \geq 0.$$

Let $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that $1/N < e\varepsilon$. Then whenever $n \geq N$ and $x \geq 0$,

$$|xe^{-nx} - 0| \leq \frac{1}{ne} \leq \frac{1}{Ne} < \varepsilon.$$

Example 2 (c.f. Section 8.1, Ex.4 & Ex.14)**.** Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{x^n}{1+x^n}.$$

Let $0 < b < 1$. Show that (f_n) converges uniformly on $[0, b]$ but not uniformly on $[0, 1]$.

Solution. The pointwise limit of (f_n) is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1/2 & \text{if } x = 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Let's show that (f_n) converges uniformly to 0 on $[0, b]$. Note that

$$\left| \frac{x^n}{1+x^n} - 0 \right| \le \frac{x^n}{1+0} = x^n \le b^n, \quad \forall n, \quad \forall x \in [0, b].$$

Let $\varepsilon > 0$. Since $0 < b < 1$, $b^n \to 0$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that $b^n < \varepsilon$ whenever $n \ge N$. It follows that

$$\left| \frac{x^n}{1+x^n} - 0 \right| \le b^n < \varepsilon, \quad \forall n \ge N, \quad \forall x \in [0, b].$$

To see that (f_n) does not converge uniformly on $[0, 1]$, we need to show that there exist $\varepsilon > 0$ such that whenever $N \in \mathbb{N}$, there exists $n \ge N$ and $x \in [0, 1]$ such that

$$\left| \frac{x^n}{1+x^n} - 0 \right| \ge \varepsilon.$$

For each $N \in \mathbb{N}$, we can choose $n = N$ and $x = 2^{-1/n}$. Then

$$\left| \frac{x^n}{1+x^n} - 0 \right| = \frac{1/2}{1+1/2} = \frac{1}{3}.$$

This shows that the convergence is not uniform on $[0, 1]$.

Remark. Is (f_n) converges uniformly on $[0, 1)$? (Investigate the above argument!)

Essentially, the above argument did the same thing in the lemma below.

Lemma (c.f. Lemma 8.1.5)**.** *A sequence of functions (f_n) defined on $A \subseteq \mathbb{R}$ does not converges to a function f uniformly on A if and only if there exists $\varepsilon > 0$, a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in A such that*

$$|f_{n_k}(x_k) - f(x_k)| \ge \varepsilon.$$

Interchange of Limits

Uniform convergence is important when we want to interchange the order of limits. The following propositions tell us that **continuity** and **integrablility** are preserved under uniform convergence.

Theorem (c.f. Theorem 8.2.2)**.** *Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ and converges uniformly to a function f on A. Suppose that each f_n is continuous on A. Then f is continuous on A.*

Remark. This theorem tells us that for each $x_0 \in A$,

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{x \to x_0} f(x) = f(x_0) = \lim_{n \to \infty} f_n(x_0) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x).$$

Theorem (c.f. Theorem 8.2.4)**.** *Let (f_n) be a sequence of functions in $\mathcal{R}[a,b]$ and converges uniformly to a function f on $[a,b]$. Then $f \in \mathcal{R}[a,b]$ and*

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b \left(\lim_{n \to \infty} f_n \right).$$

Remark. Does the same result hold for improper integral?

The preservation of derivatives is a bit different. Conditions on the derivatives of the sequences of functions are emphasised.

Proposition (c.f. Proposition 5.1 & 5.2 of Lecture Notes)**.** *Let (f_n) be a sequence of differentiable functions defined on (a,b). Suppose that there exists a point $c \in (a,b)$ such that $\lim\limits_{n \to \infty} f_n(c)$ exists and (f_n') converges uniformly to a function g on (a,b). Then*

- *(f_n) converges uniformly to a differentiable function f on (a,b); and*
- *$f' = g$ on (a,b).*

Remark. This is a stronger version of the theorem proved in the lecture. The proof is more complicated because we cannot apply the **Fundamental Theorem**. Nevertheless, these theorems both implies that for each $x_0 \in (a,b)$,

$$\lim_{x \to x_0} \lim_{n \to \infty} \frac{f_n(x) - f_n(x_0)}{x - x_0} = \lim_{n \to \infty} \lim_{x \to x_0} \frac{f_n(x) - f_n(x_0)}{x - x_0}.$$

Reading Exercise (c.f. Proposition 6.2 of Lecture Notes)**.** Read the proofs of **Dini's Theorem** to learn the "compactness arguments".

Exercises

Exercise 1 (c.f. Section 8.1, Ex.24)**.** Let (f_n) be a sequence of functions that converges uniformly to f on A and that satisfies $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in A$. If g is continuous on the interval $[-M, M]$, show that the sequence $(g \circ f_n)$ converges uniformly to $g \circ f$ on A.

Solution. We need to show that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|g(f_n(x)) - g(f(x))| < \varepsilon, \quad \forall n \geq N, \quad \forall x \in A.$$

Let $\varepsilon > 0$. Since g is continuous on $[-M, M]$, g is uniformly continuous on $[-M, M]$. i.e., there exist $\delta > 0$ such that whenever $|u - v| < \delta$ and $u, v \in [-M, M]$,

$$|g(u) - g(v)| < \varepsilon.$$

Now since (f_n) converges uniformly to f, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \delta, \quad \forall n \geq N, \quad \forall x \in A.$$

Also, take limit as $n \to \infty$ in the inequality $|f_n(x)| \leq M$, we have $|f(x)| \leq M$ for all $x \in A$. Hence whenever $n \geq N$ and $x \in A$, we have shown that $|f_n(x) - f(x)| < \delta$ and $f_n(x), f(x) \in [-M, M]$. Thus by the continuity of g,

$$|g(f_n(x)) - g(f(x))| < \varepsilon.$$

Exercise 2 (c.f. Section 8.2, Ex.7)**.** Suppose the sequence (f_n) converges uniformly to f on the set A, and suppose that each f_n is bounded on A. (i.e., for each n, there is a constant M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$.) Show that that function f is bounded on A.

Solution. We need to find a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. Take $\varepsilon = 1$. Since (f_n) converges to f uniformly, there exists $N \in \mathbb{N}$ such that

$$|f_N(x) - f(x)| < 1, \quad \forall x \in A.$$

Therefore by triangle inequality,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + M_N, \quad \forall x \in A.$$

Hence f is bounded by $M = 1 + M_N$.

Remark. In addition, we can show that the sequence of functions (f_n) is **uniformly bounded**. i.e., there exists $M > 0$ such that

$$|f_n(x)| \leq M, \quad \forall n \in \mathbb{N}, \quad \forall x \in A.$$

Exercise 3 (c.f. Section 8.2, Ex.4)**.** Suppose (f_n) is a sequence of continuous functions on an interval I that converges uniformly on I to a function f. If $(x_n) \subseteq I$ converges to $x_0 \in I$, show that

$$\lim_{n\to\infty} f_n(x_n) = f(x_0).$$

Solution. We need to show that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x_n) - f(x_0)| < \varepsilon, \quad \forall n \geq N.$$

Notice that for any $n \in \mathbb{N}$ and $x \in I$,

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|.$$

Let $\varepsilon > 0$. Since (f_n) converges to f uniformly, there exists $N_1 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \quad \forall n \geq N_1, \quad \forall x \in I.$$

On the other hand, note that f is a continuous function because it is the uniform limit of continuous functions. Hence $\lim f(x_n) = f(x_0)$. i.e., there exists $N_2 \in \mathbb{N}$ such that

$$|f(x_n) - f(x_0)| < \frac{\varepsilon}{2}, \quad \forall n \geq N_2.$$

Combine the above results, take $N = \max\{N_1, N_2\}$. Then whenever $n \geq N$,

$$|f_n(x_n) - f(x_0)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Remark. In the very beginning, we estimate $|f_n(x_n) - f(x_0)|$ by

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|.$$

What happens if we change the estimation to

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f_n(x_0)| + |f_n(x_0) - f(x_0)|?$$