Suggested Solution of Exercises on Riemann Integration

Question 1 (2018-19 Final Q2)**.** Define a function $g : [0, \pi/2] \to \mathbb{R}$ by

$$g(x) = \begin{cases} \cos^2 x, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

Find the upper and lower Riemann integrals of g over $[0, \pi/2]$. Is it Riemann integrable?

Solution. Let's find the lower and upper integrals of g.

- **Lower integral**: Let P be any partition of $[0, \pi/2]$. Note that each subinterval $[x_{i-1}, x_i]$ must containing some irrational number, so

  $$m_i(g, P) = 0, \quad \forall i = 1, ..., n.$$

  It follows that the lower sum is given by

  $$L(g, P) = \sum_{i=1}^{n} m_i(g, P)\Delta x_i = 0.$$

  Taking infimum over all partition P, the lower integral of g is given by

  $$\underline{\int_0^{\pi/2}} g = 0.$$

- **Upper integral**: Let P be any partition of $[0, \pi/2]$. Note that $\cos^2 x$ is decreasing on each subinterval $[x_{i-1}, x_i]$ and rational numbers are dense, so

  $$M_i(g, P) = \cos^2(x_{i-1}), \quad \forall i = 1, ..., n.$$

  Consider $f : [0, \pi/2] \to \mathbb{R}$ defined by $f(x) = \cos^2 x$. We also have

  $$M_i(f, P) = \cos^2(x_{i-1}), \quad \forall i = 1, ..., n.$$

  It follows that

  $$U(g, P) = \sum_{i=1}^{n} M_i(g, P)\Delta x_i = \sum_{i=1}^{n} M_i(f, P)\Delta x_i = U(f, P).$$

  Since f and g have the same upper sum over arbitrary partitions of $[0, \pi/2]$, they have the same upper integral, hence

  $$\overline{\int_0^{\pi/2}} g = \overline{\int_0^{\pi/2}} f = \int_0^{\pi/2} f = \int_0^{\pi/2} \cos^2 x dx = \int_0^{\pi/2} \frac{1 + \cos 2x}{2} dx = \frac{\pi}{4}.$$

In summary, the lower and upper intgral of g is given by

$$\underline{\int_0^{\pi/2}} g = 0 \quad \text{and} \quad \overline{\int_0^{\pi/2}} g = \frac{\pi}{4}.$$

Since they are unequal, g is **not** Riemann integrable.

Question 2 (2016-17 Midterm Q4)**.** Define a funtion f on $[0,1]$ by

$$f(x)=\begin{cases}1, & \text{if } x=1/n \text{ for some } n\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases}$$

Show that f is Riemann integrable and find $\displaystyle\int_0^1 f$.

Solution. For each natural number $N\geq 2$, define the partition P_N of $[0,1]$ by

$$P_N=\left\{0,\frac{1}{N}\pm\delta,\frac{1}{N-1}\pm\delta,...,\frac{1}{2}\pm\delta,1-\delta,1\right\},\quad \text{where } 2\delta<\frac{1}{N(N-1)}.$$

(Visualize this partition!) This ensures that

$$0<\frac{1}{N}-\delta<\frac{1}{N}+\delta<\frac{1}{N-1}-\delta<\frac{1}{N-1}+\delta<\cdots<\frac{1}{2}-\delta<\frac{1}{2}+\delta<1-\delta<1.$$

Note that each subinterval $[x_{i-1},x_i]$ must contain some numbers that cannot be represented by the reciprocal of some natural number, so

$$m_i(f,P_N)=0,\quad \forall i=1,...,n$$

On the other hand, note that we have (Why?)

$$M_i(f,P_N)=\begin{cases}1, & \text{if } i=1 \text{ or even}\\ 0, & \text{otherwise}\end{cases}.$$

Therefore, the lower sum and upper sum can be calculated by

$$L(f,P_N)=\sum_{i=1}^{n}m_i(f,P_N)\Delta x_i=0$$
$$U(f,P_N)=\sum_{i=1}^{n}M_i(f,P_N)\Delta x_i=\left(\frac{1}{N}-\delta\right)+\sum_{k=2}^{N}2\delta+\delta=\frac{1}{N}+2(N-1)\delta$$

It follows that

$$0=L(f,P_N)\leq\underline{\int_0^1} f\leq\overline{\int_0^1} f\leq U(f,P_n)<\frac{1}{N}+\frac{N-1}{N(N-1)}=\frac{2}{N}.$$

Since $n\geq 2$ is arbitrary, letting $N\to\infty$ in the above inequality, we have

$$0\leq\underline{\int_0^1} f\leq\overline{\int_0^1} f\leq 0.$$

It forces the lower and upper integral of f equal zero. Thus f is Riemann integrable with

$$\int_0^1 f=0.$$

Question 3 (2017-18 Final Q2)**.** .

(i) Define a function $f : [0, \infty) \to [0, \infty)$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in [n, n + 1/2^n) \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the improper integral $\displaystyle\int_0^\infty f(x)dx$ exists but $\displaystyle\lim_{x\to\infty} f(x)$ does not exist.

(ii) Let f be a non-negative $\mathbb{R}$-valued function defined on $[0, \infty)$. Suppose that $\displaystyle\int_0^\infty f(x)dx$ is convergent and $\displaystyle\lim_{x\to\infty} f(x) = L$. Show that $L = 0$.

Solution. .

(i) The fact that $f \in \mathcal{R}[0, T]$ for all $T > 0$ is left as an exercise. (You need to find a partition P of $[0, T]$ for each $\varepsilon > 0$ such that $U(f, P) - L(f, P) < \varepsilon$. It is tedious but the technique used is the same. You usually don't need to provide the proof for such questions if you have more to do.)

Notice that for each $n \in \mathbb{N}$,

$$\int_0^n f(x)dx = \sum_{k=0}^{n-1} \int_k^{k+1} f(x)dx = \sum_{k=1}^{n-1} \frac{1}{2^k} = 1 - \frac{1}{2^{n-1}}.$$

Fix $T > 0$ and let N to be a natural number such that $N \le T < N + 1$. Since f is non-negative, we have

$$1 - \frac{1}{2^{N-1}} = \int_0^N f(x)dx \le \int_0^T f(x)dx \le \int_0^{N+1} f(x)dx = 1 - \frac{1}{2^N}.$$

Since $N \to \infty$ as $T \to \infty$, by Squeeze Theorem, $\displaystyle\int_0^\infty f(x)dx = \lim_{T\to\infty} \int_0^T f(x)dx = 1.$

Now consider the sequences (x_n) and (y_n) defined by

$$x_n = n \quad \text{and} \quad y_n = n + \frac{1}{2^n}, \quad \forall n \in \mathbb{N}.$$

Note the both (x_n) and (y_n) diverges properly to ∞, but $f(x_n) = 1$ for all n and $f(y_n) = 0$ for all n. Hence $\displaystyle\lim_{x\to\infty} f(x)$ does not exist.

(ii) Since f is non-negative, we must have $L \ge 0$. Suppose on a contrary that $L > 0$. Then there exist $T > 0$ such that $f(x) \ge L/2$ for all $x > T$. Then for each $M > 0$, take $A > \max\{T, M\}$, we have

$$\left| \int_A^{A+1} f(x)dx \right| = \int_A^{A+1} f(x)dx \ge \frac{L}{2} > 0.$$

It follows by **Cauchy Criterion** that $\displaystyle\int_0^\infty f(x)dx$ is divergent. It is a contradiction.

Question 4 (2016-17 Final Q2)**.** Let f be a function defined by

$$f(x) = \frac{\sin x}{x}, \quad \text{for } x \geq 1.$$

(i) Show that the integral $\displaystyle\int_1^\infty f(x)dx$ is convergent.

(ii) Show that the integral $\displaystyle\int_1^\infty |f(x)|dx$ is divergent.

Solution. .

(i) Fix $T > 1$. Note that by **Integration by Parts**,

$$\int_1^T f(x)dx = \int_1^T \frac{\sin x}{x}dx = \Big[-\frac{\cos x}{x}\Big]_1^T - \int_1^T \frac{\cos x}{x^2}dx.$$

Hence it suffices to show that the improper integral $\displaystyle\int_1^\infty \frac{\cos x}{x^2}dx$ converges.

Note that for any $A_2 > A_1 > 1$, we have

$$\left|\int_{A_1}^{A_2} \frac{\cos x}{x^2}dx\right| \leq \int_{A_1}^{A_2} \frac{|\cos x|}{x^2}dx \leq \int_{A_1}^{A_2} \frac{1}{x^2}dx = \frac{1}{A_1} - \frac{1}{A_2} \leq \frac{1}{A_1}.$$

Hence for any $\varepsilon > 0$, we can choose $M > 1$ such that $1/M < \varepsilon$. Then whenever $A_2 > A_1 > M$,

$$\left|\int_{A_1}^{A_2} \frac{\cos x}{x^2}dx\right| \leq \frac{1}{A_1} < \frac{1}{M} < \varepsilon.$$

It follows by **Cauchy Criterion** that $\displaystyle\int_1^\infty \frac{\cos x}{x^2}dx$ converges.

(ii) Note that since $|f(x)|$ is non-negative, we have

$$\int_1^\infty |f(x)|dx \geq \int_\pi^{(N+1)\pi} \frac{|\sin x|}{x}dx = \sum_{k=1}^N \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}dx.$$

For each $k \in \mathbb{N}$, we can substitute $x = t + k\pi$, then

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}dx = \int_0^\pi \frac{|\sin(t+k\pi)|}{t+k\pi}dt = \int_0^\pi \frac{\sin t}{t+k\pi}dt \geq \frac{1}{(k+1)\pi}\int_0^\pi \sin t dt.$$

Write $A = \displaystyle\int_0^\pi \sin t dt$, we have

$$\int_1^\infty |f(x)|dx \geq \sum_{k=1}^N \frac{A}{(k+1)\pi} = \frac{A}{\pi}\sum_{k=1}^N \frac{1}{k+1}.$$

Since the above inequality holds for all $N \in \mathbb{N}$ and the harmonic series diverges to ∞, it follows that $\displaystyle\int_1^\infty |f(x)dx$ is also divergent.

Remark. Integration by parts is a consequence of the **Product Rule** and the **Fundamental Theorem of Calculus**. You may try to prove it as an exercise.

Question 5 (2018-19 Final Q3)**.** Let f be a continuous function on $[a, b]$ and $\varphi : [\alpha, \beta] \to \mathbb{R}$ be continuously differentiable such that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$. Show that

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

(Hint: Consider the functions $F(u) = \int_a^u f(x)dx$ and $H(t) = F(\varphi(t))$.)

Solution. Define the functions $F : [a, b] \to \mathbb{R}$ and $H : [\alpha, \beta] \to \mathbb{R}$ by

$$F(u) = \int_a^u f(x)dx \quad \text{and} \quad H(t) = F(\varphi(t)).$$

Since f is continuous, we have $F' = f$ by the **Fundamental Theorem of Calculus**. Also, by **Chain Rule**, we have $H'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t)$. It follows again by the **Fundamental Theorem of Calculus** that

$$\int_a^b f(x)dx = F(b) - F(a) = H(\beta) - H(\alpha) = \int_\alpha^\beta H'(t)dt = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

Remark. Note that this substitution theorem is a liitle bit different from the lecture notes. The assumption on φ is relaxed, but f is required to be continuous.