MATH2060B TUTORIAL 7

A Remark of FTC:

Question: What is wrong in this computation?

$$\int_{-1}^{1} 1/x^2 \, dx = \int_{-1}^{1} \frac{d}{dx}\left(-\frac{1}{x}\right) dx$$
$$= -\frac{1}{1} + \frac{1}{-1}$$
$$= -2$$

Misconception:
Attempt to use FTC on $f(x) = 1/x^2$ on $[-1, 1]$. But f is actually unbounded!
It is meaningless to discuss it integrability!
This is actually an "improper integral", we should handle it with more care.

Riemann Sums:

Definition: Let f be a function defined on $[a, b]$ and P be a partition of $[a, b]$.
(not necessarily bounded) For each $i = 1, \ldots, n$, let $\xi_i \in [x_{i-1}, x_i]$.

* Denote the Riemann sum of f by:

$$R(f, P, \{\xi_i\}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

* The Riemann sum $R(f, P, \{\xi_i\})$ is said to converge to a number A as $\|P\| \to 0$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that whenever $\|P\| < \delta$ and $\xi_i \in [x_{i-1}, x_i]$,
$$|R(f, P, \{\xi_i\}) - A| < \varepsilon$$

Remark: If the Riemann sum of f converges, then f is automatically bounded.
The limit of the Riemann sum depends only on f, but not P or $\{\xi_i\}$.

Theorem: Let f be a function defined on $[a, b]$. Then $f \in R[a, b]$ if and only if the Riemann sum of f is convergent. In this case,

$$\lim_{\|P\| \to 0} R(f, P, \{\xi_i\}) = \int_a^b f$$

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Application: Evaluate limits

Example 1: Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$

Solution: By definition, we need to show that $\forall\, \varepsilon > 0$, there exists N such that

$$\left| \sum_{k=1}^{n} \frac{n}{n^2+k^2} - L \right| < \varepsilon \qquad \forall\, n \geq N$$

Note that for each natural number n,

$$\sum_{k=1}^{n} \frac{n}{n^2+k^2} = \sum_{k=1}^{n} \frac{n^2}{n^2+k^2}\,\frac{1}{n} = \sum_{k=1}^{n} \underbrace{\frac{1}{1+(k/n)^2}}_{f(\xi_k)}\,\underbrace{\frac{1}{n}}_{\Delta x_k}$$

Let $f(x) = 1/(1+x^2)$ for $x \in [0, 1]$, it has an anti-derivative $F(x) = \arctan x$. Hence

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^1 f(x)\,dx = F(1) - F(0) = \arctan(1) - \arctan(0) = \frac{\pi}{4}$$

Let $\varepsilon > 0$, consider the Riemann sum of f, there exists $\delta > 0$ such that whenever P is a partition of $[0, 1]$ with $\|P\| < \delta$ and $\xi_k \in [x_{k-1}, x_k]$,

$$\left| \sum_{k=1}^{n} f(\xi_k)\Delta x_k - \frac{\pi}{4} \right| < \varepsilon \qquad (*)$$

Let N be a natural number such that $1/N < \delta$. Then whenever $n \geq N$, consider the partition P_n of $[0, 1]$ defined by $P_n = \{0, 1/n, 2/n, \ldots, 1\}$ and $\xi_k = 1/k$.
Then $\| P_n \| = 1/n \leq 1/N < \delta$ and $\xi_k \in [(k-1)/n, k/n]$. Thus by $(*)$,

$$\begin{aligned}\left| \sum_{k=1}^{n} \frac{n}{n^2+k^2} - \frac{\pi}{4} \right| &= \left| \sum_{k=1}^{n} \frac{1}{1+(k/n)^2}\,\frac{1}{n} - \frac{\pi}{4} \right| \\ &= \left| \sum_{k=1}^{n} f(\xi_k)\Delta x_k - \frac{\pi}{4} \right| \\ &< \varepsilon\end{aligned}$$

It follows that the limit is $\pi/4$. #

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Example 2: Evaluate $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$

Solution: We first consider the limit of the logarithm of the expression.

$$\ln\left(\frac{n!}{n^n}\right)^{1/n} = \frac{1}{n}\ln\frac{n!}{n^n} = \frac{1}{n}\ln\left(\frac{1}{n}\,\frac{2}{n}\cdots\frac{n}{n}\right) = \sum_{k=1}^{n}\ln(k/n)\,\frac{1}{n}$$

Hence

$$\lim_{n\to\infty}\ln\left(\frac{n!}{n^n}\right)^{1/n} = \lim_{n\to\infty}\sum_{k=1}^{n}\ln(k/n)\,\frac{1}{n}$$

$$= \int_0^1 \ln x\,dx \qquad \text{(This is actually an improper integral)}$$

$$= x\ln x - x\Big]_0^1$$

$$= -1$$

It follows that the limit is e^{-1}. #

Substitution Theorem:

7.3.8 Substitution Theorem *Let $J := [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$, then*

(5) $$\int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx.$$

Remark: The hypothesis is restrictive. It simplifies the proof and ensure some integrability. Please refer to the proof of the theorem in the notes. It is better to understand the idea of the proof first. Since it takes time to be fully presented.

Exercises:

1. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $c > 0$, define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := \int_{x-c}^{x+c} f(t)dt$. Show that g is differentiable on $\mathbb{R}$ and find $g'(x)$.

Solution: Since f is continuous, f has an anti-derivative F by FTC.
Also by FTC, $g(x) = F(x+c) - F(x-c)$.
To show that g is differentiable and find its derivative, we calculate

$$\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \left[\frac{F(x+h+c) - F(x+h-c)}{h} - \frac{F(x+c) - F(x-c)}{h} \right]$$

$$= \lim_{h\to 0} \left[\frac{F(x+h+c) - F(x+c)}{h} - \frac{F(x+h-c) - F(x-c)}{h} \right]$$

$$= \lim_{h\to 0} \frac{F(x+h+c) - F(x+c)}{h} - \lim_{h\to 0} \frac{F(x+h-c) - F(x-c)}{h}$$

$$= F'(x+c) - F'(x-c)$$

$$= f(x+c) - f(x-c)$$

It follows that $g(x)$ is differentiable and $g'(x) = f(x+c) - f(x-c)$. #

2. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

Solution: Using the same technique, write F an anti-derivative of f. Then

$$F(x) - F(0) = F(1) - F(x)$$

i.e., $F(x) = (F(0) + F(1))/2$ is a constant.
It follows that $f(x) = F'(x) = 0$ for all $x \in [0, 1]$. #

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3. Use the Substitution Theorem 7.3.8 to evaluate the following integrals.

(a) $\int_0^1 t\sqrt{1+t^2}\,dt$,

(b) $\int_0^2 t^2(1+t^3)^{-1/2}\,dt = 4/3$,

(c) $\int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}}\,dt$,

(d) $\int_1^4 \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt = 2(\sin 2 - \sin 1)$.

Solution: Let's do (a) and (c) as you are given the answers for (b) and (d).

(a) Write $f(x) = \sqrt{x}$ and $\phi(t) = 1 + t^2$. Note that Φ is continuously differentiable and strictly increasing on $[0, 1]$. It follows that

$$\int_0^1 t\sqrt{1+t^2}\,dt = \frac{1}{2}\int_0^1 f(\phi(t))\,\phi'(t)\,dt$$

$$= \frac{1}{2}\int_{\phi(0)}^{\phi(1)} f(x)dx \qquad \text{used substitution theorem here}$$

$$= \frac{1}{2}\int_1^2 \sqrt{x}\,dx$$

$$= \frac{1}{2}\,\frac{2}{3}\,x^{3/2}\Big]_1^2 \qquad \text{Find an anti-derivative of } f$$

$$= \frac{1}{3}(2\sqrt{2}-1)$$

(c) Write $f(x) = \sqrt{x}$ and $\phi(t) = 1 + \sqrt{t}$. Note that Φ is continuously differentiable and strictly increasing on $[1, 4]$. It follows that

$$\int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}}\,dt = 2\int_1^4 f(\phi(t))\,\phi'(t)\,dt$$

$$= 2\int_{\phi(0)}^{\phi(4)} f(x)dx$$

$$= 2\int_1^3 \sqrt{x}\,dx$$

$$= \frac{4}{3}(3\sqrt{3}-1)$$

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