Series of Functions

Similar to series of real numbers, we can define series of functions and the concept of absolute convergence (because of series) and uniform convergence (because of functions).

Definition (c.f. Definition 9.4.1)**.** Let (f_n) be a sequence of real-valued functions defined on $A \subseteq \mathbb{R}$. The series $\sum f_n$ is said to **converge (pointwisely)** to a function f on A if its **partial sum**

$$s_n = \sum_{k=1}^{n} f_k$$

is convergent to f. i.e., for each $x \in A$ and for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{n} f_k(x) - f(x) \right| < \varepsilon, \quad \forall n \geq N.$$

In this case, we denote

$$f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x) \quad \text{or} \quad f = \sum_{n=1}^{\infty} f_n.$$

The series is said to **converge uniformly** to f on A if its partial sum s_n converges uniformly to f. i.e., for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{n} f_k(x) - f(x) \right| < \varepsilon, \quad \forall n \geq N \text{ and } \forall x \in A.$$

Since there are nothing new but only rephrasing old definitions, previous results on uniform convergence of sequence of functions apply on series of function.

Exercise 1 (c.f. Theorem 9.4.2, 9.4.3 & 9.4.4)**.** Formulate the theorems on interchanging limits of uniformly convergent series of functions.

Cauchy Criterion (c.f. 9.4.5)**.** *Let (f_n) be a sequence of functions defined on A. The series $\sum f_n$ is uniformly convergent on A if and only if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| < \varepsilon, \quad \forall n \geq N, \quad \forall p \in \mathbb{N}, \quad \forall x \in A.$$

The most important test of uniform convergence of series of functions is the following:

Weierstrass M-Test (c.f. 9.4.6)**.** *Let (f_n) be a sequence of real-valued functions defined on $A \subseteq \mathbb{R}$ and (M_n) be a sequence of positive real numbers such that $|f_n(x)| \leq M_n$ for $x \in A$, $n \in \mathbb{N}$. If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.*

Example 1 (c.f. Section 9.4, Ex.1)**.** Discuss the convergence of the following series of functions.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$, where $x \in \mathbb{R}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 x^2}$, where $x \neq 0$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^n + 1}$, where $x \geq 0$.

Solution. .

(a) We can directly apply **Weierstrass M-Test**. Notice that

$$|f_n(x)| = \frac{|\cos nx|}{n^2} \leq \frac{1}{n^2}, \quad \forall n \in \mathbb{N}, \quad \forall x \in \mathbb{R}.$$

Hence by Weierstrass M-Test, the series is uniformly convergent.

(b) Let's discuss pointwise convergence first. Obviously, we have

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 x^2} = \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6x^2}, \quad \forall x \neq 0.$$

Hence the series converges pointwisely. Since the series is unbounded near 0, it is natural to think that it is not uniformly convergent on its domain of definition. However, if we fix $a > 0$, we can show that the series is uniformly convergent for $|x| \geq a$:

$$|f_n(x)| = \frac{1}{n^2 x^2} \leq \frac{1}{n^2 a^2}, \quad \forall n \in \mathbb{N}, \quad |x| \geq a.$$

Hence by **M-Test**, the series is uniformly convergent on $(-\infty, -a] \cup [a, \infty)$.

(c) First of all, if $0 \leq x \leq 1$, then

$$\lim_{n\to\infty} \frac{1}{x^n + 1} \neq 0.$$

Hence by **n-th Term Test**, the series is divergent. On the other hand, if $x > 1$, then

$$|f_n(x)| = \frac{1}{x^n + 1} \leq \frac{1}{x^n} = \left(\frac{1}{x}\right)^n, \quad \forall n, \in \mathbb{N}, \quad \forall x > 1.$$

Hence by the **Comparison Test**, the series is converge pointwisely for $x > 1$. Similar to the above example, if we fix $a > 1$, then the series converges uniformly on $[a, \infty)$. Let's see why it is not uniformly convergent on $(1, \infty)$ by using **Cauchy Criterion**. For each $N \in \mathbb{N}$, take $n = N$, $p = 1$ and $x = 2^{1/(n+1)} \in (1, \infty)$. Then

$$|f_{n+1}(x) + \cdots + f_{n+p}(x)| = \frac{1}{x^{n+1} + 1} = \frac{1}{2 + 1} = \frac{1}{3}.$$

Hence the series is not uniformly convergent.

Power Series

Power series is a typical example of series of function and is of great interest.

Definition (c.f. Definition 9.4.7)**.** Let (a_n) be a sequence of real numbers and $c \in \mathbb{R}$. A (formal) **power series** centered at c is defined as

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

Denote $\operatorname{dom}(f)$ as the set of $x \in \mathbb{R}$ for which $f(x)$ is convergent.

Remark. Notice that $c \in \operatorname{dom}(f)$, so $\operatorname{dom}(f)$ must be non-empty. By a translation, we can restrict the attention on power series that centered at 0. i.e., the power series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Definition (c.f. Definition 9.4.8)**.** Let f be a power series centered at 0. The **radius of convergence** R of f is defined to be the supremum of $\operatorname{dom}(f)$, i.e.,

$$R = \begin{cases} 0, & \text{if } \operatorname{dom}(f) = \{0\}, \\ \sup \operatorname{dom}(f), & \text{if } \operatorname{dom}(f) \text{ is bounded,} \\ \infty, & \text{if } \operatorname{dom}(f) \text{ is unbounded.} \end{cases}$$

The following theorem describes a nice behaviour of the domains of power series.

Cauchy-Hadamard Theorem (c.f. 9.4.9)**.** *Let R be the radius of convergence of a power series f centered at 0. Then f converges absolutely for $|x| < R$ and diverges for $|x| > R$.*

Theorem (c.f. Theorem 9.4.10)**.** *Let R be the radius of convergence of a power series f centered at 0. Let $0 < \eta < R$. Then f converges uniformly on $[-\eta, \eta]$.*

Remark. The **Cauchy Hadamard Theorem** tells us that $\operatorname{dom}(f)$ must be an interval with endpoints $-R$ and R. i.e., $\operatorname{dom}(f)$ takes one of the following forms:

$$\{0\}, \quad (-\infty, \infty), \quad (-R, R), \quad [-R, R], \quad (-R, R], \quad [-R, R).$$

However, the behaviour of the power series at the endpoints $-R$ and R is unclear.
Also, the radius of convergence is determined by the coefficients a_n. We can find it by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

with the convention that $1/0 = \infty$ and $1/\infty = 0$ and provided that the limit exists.

Since power series fulfils some uniformly convergent property, the theorems on interchanging limits applies.

Theorem (c.f. Theorem 9.4.11 & 9.4.12)**.** *Let $f(x) = \sum a_n x^n$ be a power series centered at 0 with radius of convergence R. Then*

(i) f is continuous on $\mathrm{dom}(f)$.

(ii) f can be integrated term-by-term. i.e., for any $x \in (-R, R)$,

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} a_n \int_0^x t^n dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

(iii) f can be differentiated term-by-term. i.e., for any $x \in (-R, R)$,

$$f'(x) = \sum_{n=0}^{\infty} \Big(a_n x^n\Big)' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Moreover, the integrated and differentiated series have the same radius convergence R as f.

Remark. We can see from the above theorem that a power series is infinitely differentiable. If we consider $f^{(k)}(0)$ for $k = 0, 1, 2, ...$, we see that the coefficients a_n are given by

$$a_n = \frac{1}{n!} f^{(n)}(0), \quad \forall n = 0, 1, 2, ...$$

Compare it with the Taylor series of f for more mathematical insights.

Example 2. Observe the following examples of powers series:

(a) The power series expansion of e^x is given by

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots, \quad x \in \mathbb{R}.$$

(b) The power series expansion of $\sin x$ is given by

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots, \quad x \in \mathbb{R}.$$

(c) The power series expansion of $\cos x$ is given by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+1} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots, \quad x \in \mathbb{R}.$$

(d) The power series expansion of $1/(1+x)$ is given by

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots, \quad |x| < 1.$$

Example 3 (c.f. Section 9.4, Ex.18)**.** Show that if $|x| < 1$, then

$$\arcsin x = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \cdot \frac{1}{2n+1} x^{2n+1}$$

Solution. Notice that the derivative of $\arcsin x$ is given by

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}.$$

Hence we can find the power series of its derivative and integrate term-by-term to retrieve the power series of $\arcsin x$. Recall the **Generalized Binomial Theorem**: If $\alpha \in \mathbb{R}$, then

$$(1+t)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n, \quad |t| < 1.$$

Since $|x| < 1$, we have $|(-x^2)| < 1$. We can apply the theorem for $\alpha = -1/2$ and $t = -x^2$.

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n}.$$

Now we compute

$$\binom{-1/2}{n} = \frac{(-\frac{1}{2}) \cdot (-\frac{1}{2} - 1) \cdots [-\frac{1}{2} - (n-1)]}{n!} = (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}.$$

It follows that

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} x^{2n}.$$

After intergrating term-by-term, we have

$$\arcsin x = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \cdot \frac{1}{2n+1} x^{2n+1}.$$