Counter-examples of Sequence of Functions

The sequence of functions (x^n) defined on $[0,1]$ is a very famous example that does not converge uniformly to its pointwise limit. Let's see more special examples.

Example 1 (c.f. Example 8.2.1(c))**.** For each positive integer $n \geq 2$, define the function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n^2 x, & \text{if } 0 \leq x < 1/n, \\ -n^2(x - 2/n), & \text{if } 1/n \leq x < 2/n, \\ 0, & \text{if } 2/n \leq x \leq 1. \end{cases}$$

(Draw the graphs of the functions!) Notice that $f_n(0) = 0$ for all $n \geq 2$. Therefore

$$\lim_{n\to\infty} f_n(0) = 0.$$

On the other hand, if $0 < x \leq 1$, we have $x \geq 2/n$ for sufficiently large n. i.e., $f_n(x) = 0$ for such n's. Therefore

$$\lim_{n\to\infty} f_n(x) = 0.$$

It follows that (f_n) converges pointwisely to 0. Now consider the integral of f_n over $[0,1]$. By looking at their graphs, we have

$$\int_0^1 f_n(x)dx = 1, \quad \forall n \geq 2.$$

It follows that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1 \neq 0 = \int_0^1 \left(\lim_{n\to\infty} f(x) \right) dx.$$

We see that (f_n) does not converge uniformly to 0 because the integral is not preserved.

Example 2 (Section 8.2, Ex.16)**.** Let (r_n) be an enumeration of all rational numbers in $[0,1]$. For each n, define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1, & \text{if } x = r_i \text{ for some } i = 1, ..., n, \\ 0, & \text{otherwise.} \end{cases}$$

If $x \in \mathbb{Q}$, there exists some N such that $x = r_N$. Therefore $f_n(x) = 1$ for all $n \geq N$. Hence

$$\lim_{n\to\infty} f_n(x) = 1.$$

On the other hand, if $x \notin \mathbb{Q}$, then $f_n(x) = 0$ for all n. It follows that (f_n) converges to the **Dirichlet's function**, which is not even integrable in this case.

Example 3. For each n, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} |x|, & \text{if } |x| > 1/n, \\ (n^2x^2+1)/2n, & \text{if } |x| \leq 1/n. \end{cases}$$

(Draw the graphs of the functions!) If $|x| > 1/n$, then $f_n(x) = |x|$ for all n; If $|x| \leq 1/n$,

$$|f_n(x) - |x|| = \left| \frac{n^2x^2 - 2n|x| + 1}{2n} \right| = \frac{(1-n|x|)^2}{2n} \leq \frac{1}{2n}, \quad \forall n.$$

Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $1/N < 2\varepsilon$. Then whenever $n \geq N$ and $x \in \mathbb{R}$,

$$\Big| f_n(x) - |x| \Big| \leq \frac{1}{2n} \leq \frac{1}{2N} < \varepsilon.$$

This shows that (f_n) **converges uniformly** to $|x|$. On the other hand, notice that each f_n is differentiable on $\mathbb{R}$ (**Exercise!**). We see that (f_n) is a sequence of differentiable functions that converges uniformly to a function that is not differentiable! It is because the derivatives (f_n') is **NOT** uniformly convergent. In fact, the derivatives f_n' is given by

$$f_n'(x) = \begin{cases} -1, & \text{if } x < -1/n, \\ nx, & \text{if } |x| \leq 1/n, \\ 1, & \text{if } x > 1/n. \end{cases}$$

with pointwise limit given by

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Series of Real Numbers

Definition (c.f. Definition 3.7.1)**.** Let (x_n) be a sequence of real numbers. Denote s_n the **n-th partial sum** of (x_n) given by

$$s_n = x_1 + x_2 + \cdots + x_n = \sum_{k=1}^{n} x_k.$$

The **(infinite) series** generated by (x_n) is said to converge if the limit of (s_n) exists. In this case, we denote

$$\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k.$$

Remark. The order of x_n is important to determine the limit.

Example 4 (c.f. Section 3.7, Ex.3(b))**.** Show that if $\alpha > 0$, then

$$\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha}.$$

Solution. Notice that by partial fraction,

$$\frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha+n} - \frac{1}{\alpha+n+1}, \quad \forall n.$$

Hence for every n, we have

$$\sum_{k=0}^{n} \frac{1}{(\alpha+k)(\alpha+k+1)} = \sum_{k=0}^{n} \left(\frac{1}{\alpha+n} - \frac{1}{\alpha+n+1} \right) = \frac{1}{\alpha+0} - \frac{1}{\alpha+n+1}.$$

Taking limit as $n \to \infty$, we have

$$\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \lim_{n\to\infty} \left(\frac{1}{\alpha+0} - \frac{1}{\alpha+n+1} \right) = \frac{1}{\alpha}.$$

In the study of series, ones interested in determining whether a given series is convergent or not. There are many tests of convergence invented. The following are some commonly seen tests.

The n-th term Test (c.f. 3.7.3)**.** *If the series* $\sum x_n$ *converges, then* $\lim x_n = 0$.

Cauchy Criterion for Series (c.f. 3.7.4)**.** *The series* $\sum x_n$ *converges if and only if for every* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that*

$$|x_n + x_{n+1} + \cdots + x_{n+p}| < \varepsilon, \forall n \geq N, \quad \forall p \in \mathbb{N}.$$

Theorem (c.f. Theorem 3.7.5)**.** *Let* (x_n) *be a sequence of* ***non-negative*** *real numbers. Then the series* $\sum x_n$ *converges if and only if the sequence of partial sums* (s_n) *is bounded. In this case,*

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n = \sup_n s_n$$

The following proposition is very important, most of the proof of tests rely on it.

Comparison Test (c.f. 3.7.7)**.** *Let* (x_n) *and* (y_n) *be sequences of real numbers. Suppose there exists* $N \in \mathbb{N}$ *such that*

$$0 \leq x_n \leq y_n, \quad \forall n \geq N.$$

Then

(a) the convergence of $\sum y_n$ *implies the convergence of* $\sum x_n$.

(b) the divergence of $\sum x_n$ *implies the divergence of* $\sum y_n$.

Exercises

Exercise 1 (c.f. Section 3.7, Ex.14)**.** Let (a_n) be a sequence of positive real numbers and let

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad \forall n.$$

Show that $\sum b_n$ must be divergent.

Solution. Notice that

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \frac{a_1}{n} \geq 0, \quad \forall n.$$

Since the harmonic series is divergent. Hence $\sum b_n$ is divergent by **Comparison Test**.

Exercise 2 (c.f. Section 3.7, Ex.9)**.** Determine whether the following series is convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \cos n$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

Solution. .

(a) It is divergent. By the **n-th Term Test**, it suffices to show that $(\cos n)$ does not converge to 0. Suppose on a contrary that $\cos n \to 0$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that

$$|\cos n| < 0.1, \quad \forall n \geq N.$$

Now consider $\cos 2N$, we have $\cos 2N = 2\cos^2 N - 1$. This motivates us to estimate $2\cos^2 N - 1$. By the above condition when $n = N$, we have

$$-0.1 < \cos N < 0.1.$$

Hence

$$-1 \leq 2\cos^2 N - 1 < -0.98.$$

It follows that $|\cos 2N| > 0.98$, which contradicts to the fact that $|\cos 2N| < 0.1$.

(b) It is convergent. Notice that

$$\left|\frac{\cos n}{n^2}\right| = \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}, \quad \forall n.$$

Since $\sum 1/n^2$ is convergent, $\sum |\cos n/n^2|$ is also convergent by **Comparison Test**. Since absolute convergence implies convergence. It follows that the series $\sum \cos n/n^2$ is convergent.

Remark. We will discuss **absolute convergence** and **conditional convergence** next time. Notice that although we know that the serie is convergent in this exercise, it is not easy to find its limit.