Exercises on Riemann Integration

Please have a look at the following problems. You may as well work on it as a revision. I will present the suggested solutions in the tutorial and post them afterwards.

Question 1 (2018-19 Final Q2)**.** Define a function $g : [0, \pi/2] \to \mathbb{R}$ by

$$g(x) = \begin{cases} \cos^2 x, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

Find the upper and lower Riemann integrals of g over $[0, \pi/2]$. Is it Riemann integrable?

Question 2 (2016-17 Midterm Q4)**.** Define a funtion f on $[0, 1]$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is Riemann integrable and find $\displaystyle\int_0^1 f$.

Question 3 (2017-18 Final Q2)**.** .

(i) Define a function $f : [0, \infty) \to [0, \infty)$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in [n, n + 1/2^n) \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the improper integral $\displaystyle\int_0^\infty f(x)dx$ exists but $\displaystyle\lim_{x\to\infty} f(x)$ does not exist.

(ii) Let f be a non-negative $\mathbb{R}$-valued function defined on $[0, \infty)$. Suppose that $\displaystyle\int_0^\infty f(x)dx$ is convergent and $\displaystyle\lim_{x\to\infty} f(x) = L$. Show that $L = 0$.

Question 4 (2016-17 Final Q2)**.** Let f be a function defined by

$$f(x) = \frac{\sin x}{x}, \quad \text{for } x \geq 1.$$

(i) Show that the integral $\displaystyle\int_1^\infty f(x)dx$ is convergent.

(ii) Show that the integral $\displaystyle\int_1^\infty |f(x)|dx$ is divergent.

Question 5 (2018-19 Final Q3)**.** Let f be a continuous function on $[a, b]$ and $\varphi : [\alpha, \beta] \to \mathbb{R}$ be continuously differentiable such that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$. Show that

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

(Hint: Consider the functions $F(u) = \int_a^u f(x)dx$ and $H(t) = F(\varphi(t))$.)