Question 1. Let $g : [c, d] \to \mathbb{R}$ be a continuous function. Prove or disprove the following statements.

(a) Let $f : [a, b] \to [c, d]$ be a function. If f is Riemann integrable over $[a, b]$, then the composition $g \circ f$ is Riemann integrable over $[a, b]$.

(b) Let $h : (a, b] \to [c, d]$ be a function. If the improper integral of h over $(a, b]$ exists, then the improper integral of $g \circ h$ over $(a, b]$ exists.

(c) Let $w : [a, \infty) \to [c, d]$ be a function. If the improper integral of w over $[a, \infty)$ exists, then the improper integral of $g \circ w$ over $[a, \infty)$ exists.

Solution. .

(a) **This statement is true.** Let $\varepsilon > 0$. Notice that g is bounded and uniformly continuous on $[c, d]$. Hence there exist $M > 0$ and $\delta > 0$ such that $|g| \le M$ on $[c, d]$ and

$$|g(s) - g(t)| < \frac{\varepsilon}{2(b-a)}, \quad \text{whenever } |s - t| < \delta \text{ and } s, t \in [c, d]. \tag{1}$$

Since f is Riemann integrable over $[a, b]$, there is a partition P of $[a, b]$ such that

$$\sum_{i=1}^{n} \omega_i(f, P)\Delta x_i < \frac{\delta\varepsilon}{4M}.$$

Divide the indices $i = 1, ..., n$ into two disjoint sets:

$$I = \{i : \omega_i(f, P) < \delta\} \quad \text{and} \quad J = \{i : \omega_i(f, P) \ge \delta\}$$

If $i \in I$, then whenever $x, y \in [x_{i-1}, x_i]$, $|f(x) - f(y)| \le \omega_i(f, P) < \delta$. Hence by (1),

$$|g \circ f(x) - g \circ f(y)| = |g(f(x)) - g(f(y))| < \frac{\varepsilon}{2(b-a)}.$$

Since $x, y \in [x_{i-1}, x_i]$ are arbitrary, we have $\omega_i(g \circ f, P) \le \frac{\varepsilon}{2(b-a)}$. Therefore

$$\sum_{i\in I} \omega_i(g \circ f, P)\Delta x_i \le \frac{\varepsilon}{2(b-a)} \sum_{i\in I} \Delta x_i \le \frac{\varepsilon}{2(b-a)}(b - a) = \frac{\varepsilon}{2}. \tag{2}$$

On the other hand, if $i \in J$, then

$$\delta \sum_{i\in J} \Delta x_i \le \sum_{i\in J} \omega_i(f, P)\Delta x_i \le \sum_{i=1}^{n} \omega_i(f, P)\Delta x_i < \frac{\delta\varepsilon}{4M}.$$

It follows that

$$\sum_{i\in J} \Delta x_i < \frac{\varepsilon}{4M}.$$

Since $|g| \leq M$, we have $\omega_i(g \circ f, P) \leq 2M$ for each i. Therefore

$$\sum_{i \in J} \omega_i(g \circ f, P)\Delta x_i \leq 2M \sum_{i \in J} \Delta x_i < 2M \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}. \tag{3}$$

Finally, it follows by (2) and (3) that

$$\sum_{i=1}^{n} \omega_i(g \circ f, P)\Delta x_i = \sum_{i \in I} \omega_i(g \circ f, P)\Delta x_i + \sum_{i \in J} \omega_i(g \circ f, P)\Delta x_i < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $g \circ f$ is Riemann integrable over $[a, b]$.

(b) **This statement is true.** Note that for any $T \in (a, b)$, h is Riemann integrable over $[T, b]$ because the improper integral of h exists. By (a), it follows that $g \circ h$ is Riemann integrable over $[T, b]$. It suffices to show that the following limit exists:

$$\lim_{T \to a^+} \int_T^b g \circ h(x) dx \tag{4}$$

Let $\varepsilon > 0$ and $M > 0$ be a bound of g. i.e., $|g| \leq M$ on $[c, d]$. Take $\delta > 0$ such that

$$\delta < \frac{\varepsilon}{2M} \quad \text{and} \quad a + \delta \leq b.$$

Then whenever $a < A_1 < A_2 < a + \delta$, we have $A_2 - A_1 < \delta$. Hence

$$\left| \int_{A_1}^{A_2} g \circ h(x) dx \right| \leq \int_{A_1}^{A_2} |g \circ h(x)| dx \leq (A_2 - A_1) \cdot 2M < \delta \cdot 2M < \varepsilon.$$

It follows by **Cauchy Criterion** that the limit in (4) exists.

(c) **This statement is false.** Consider $g = 1$ to be the constant function on $[c, d]$ and w to be any function that satisfies the given condition. Then $g \circ w = 1$ on $[a, \infty]$, so its improper integral diverges properly to ∞.

Question 2. Recall the following definitions:

- A function $s : [a, b] \to \mathbb{R}$ is called a *step function* if there is a partition P of $[a, b]$ such that the restriction of s on each open subintervals (x_{i-1}, x_i) is constant.
- For each non-empty subset A of $[0, 1]$, denote
$$m(A) = \inf \sum_{i=1}^{n}(b_i - a_i),$$
where the infinum is taken over all finite collection of open intevals $\{(a_i, b_i)\}_{i=1}^{n}$ such that $A \subseteq \bigcup_{i=1}^{n}(a_i, b_i)$.

Let f be a bounded function defined on $[0, 1]$.

(a) Show that if f is Riemann integrable over $[0, 1]$, then for any $\varepsilon > 0$, there is a step function s on $[0, 1]$ such that
$$m(\{x \in [0, 1] : |f(x) - s(x)| \geq \varepsilon\}) < \varepsilon.$$

(b) Does the converse in (a) holds?

Solution. .

(a) Let $\varepsilon > 0$. Since $f \in \mathcal{R}[0, 1]$, there exists a partition P of $[0, 1]$ such that
$$\sum_{i=1}^{n} \omega_i(f, P)\Delta x_i < \varepsilon^2.$$
Define the step function $s : [0, 1] \to \mathbb{R}$ by
$$s(x) = \begin{cases} m_i(f, P), & \text{if } x \in (x_{i-1}, x_i), \\ f(x_i), & \text{if } x = x_i. \end{cases}$$
Write $A = \{x \in [0, 1] : |f(x) - s(x)| \geq \varepsilon\}$, we claim that the step function s satisfies
$$m(A) < \varepsilon.$$
Define $I = \{i : (x_{i-1}, x_i) \cap A \neq \emptyset\}$. Notice that $x_i \notin A$ for all i because
$$|f(x_i) - s(x_i)| = 0 < \varepsilon.$$
Therefore $A \subseteq \bigcup_{i \in I}(x_{i-1}, x_i)$. For each $i \in I$, pick any $y_i \in (x_{i-1}, x_i) \cap A$. Then
$$\varepsilon \leq |f(y_i) - s(y_i)| \leq \omega_i(f, P).$$
Hence
$$\varepsilon \sum_{i \in I}(x_i - x_{i-1}) \leq \sum_{i \in I} \omega_i(f, P)\Delta x_i \leq \sum_{i=1}^{n} \omega_i(f, P)\Delta x_i < \varepsilon^2.$$
It follows that
$$m(A) \leq \sum_{i \in I}(x_i - x_{i-1}) < \varepsilon.$$

(b) **The converse in (a) holds.** Let M be a bound of f and $\varepsilon > 0$, by assumption, there exists a step function s such that

$$m(\{x \in [0,1] : |f(x) - s(x)| \geq \varepsilon\}) < \varepsilon.$$

Let Q be a partition of $[0,1]$ such that s is constant on each open subintervals (x_{i-1}, x_i). We can further assume that $s(x_i) = f(x_i)$ as in (a) because it does not affect the condition above. Write $A = \{x \in [0,1] : |f(x) - s(x)| \geq \varepsilon\}$. By definition of $m(A)$, there is a finite collection open intervals $\{(a_i, b_i)\}_{i=1}^m$ such that

$$A \subseteq \bigcup_{i=1}^m (a_i, b_i) \quad \text{and} \quad \sum_{i=1}^m (b_i - a_i) < \varepsilon.$$

Define the partition P of $[0,1]$ by

$$P = Q \cup (\{a_1, b_1, a_2, b_2, ..., a_m, b_m\} \cap [0,1]),$$

and denote $[y_{j-1}, y_j]$ to be its subintervals. Divide the indices j into two disjoint sets:

$$I = \{j : [y_{j-1}, y_j] \cap A = \emptyset\} \quad \text{and} \quad J = \{j : [y_{j-1}, y_j] \cap A \neq \emptyset\}.$$

- If $j \in I$, note that we must have $[y_{j-1}, y_j] \subseteq [x_{i-1}, x_i]$ for some i. Since s is constant on (x_{i-1}, x_i), then whenever $x, y \in [y_{j-1}, y_j]$,

  $$|f(x) - f(y)| \leq |f(x) - s(x)| + |s(y) - f(y)| < \varepsilon + \varepsilon = 2\varepsilon.$$

  Taking supremum over x, y, it follows that $\omega_j(f, P) \leq 2\varepsilon$. Hence

  $$\sum_{j \in I} \omega_j(f, P) \Delta y_j \leq 2\varepsilon \sum_{j=1}^n \Delta y_j = 2\varepsilon.$$

- Note that we must have $\bigcup_{j \in J} [y_{j-1}, y_j] \subseteq \bigcup_{i=1}^m [a_i, b_i]$ for some i. Then

  $$\sum_{j \in J} \Delta y_j \leq \sum_{i=1}^n (b_i - a_i) < \varepsilon.$$

  Hence

  $$\sum_{j \in J} \omega_j(f, P) \Delta y_j \leq 2M \sum_{j \in J} \Delta y_j < 2M\varepsilon.$$

Thus $f \in \mathcal{R}[0,1]$ because

$$\sum_{j=1}^n \omega_j(f, P) \Delta y_j = \sum_{j \in I} \omega_j(f, P) \Delta y_j + \sum_{j \in J} \omega_j(f, P) \Delta y_j < 2\varepsilon + 2M\varepsilon = (2M+2)\varepsilon.$$

Remark. We should replace ε by $\dfrac{\varepsilon}{2M+2}$ in the beginning.