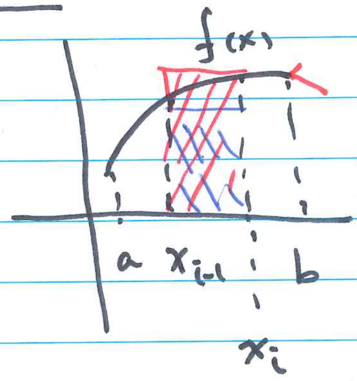
24/2/2020 ①

Recall: Notation

- $f : [a,b] \longrightarrow \mathbb{R}$ is a bounded funct
  $\Rightarrow$ $m \le f(x) \le M$. $\forall x \in [a,b]$

- partition P on $[a,b]$
  $P : a = x_0 < \cdots < x_n = b$.

  $\Delta x_i = x_i - x_{i-1}$



$$m \le \begin{matrix} M_i(f,P) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \\ m_i(f,P) = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} \end{matrix} \le M, \ \forall i$$

$$U(f,P) = \sum_{i=1}^{n} M_i(f,P)\Delta x_i \quad \text{(Upper Sum)}$$

$$L(f,P) = \sum_{i=1}^{n} m_i(f,P)\Delta x_i \quad \text{(Lower Sum)}$$

- $$\boxed{m(b-a) \le L(f,P) \le U(f,P) \le M(b-a)} \quad \forall P$$

Large

(Go back : $R(f, P, \{\xi_i\}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$

$\xi_i \in [x_{i-1}, x_i]$

$\boxed{\text{What's an area ?}}$

We have ask :

$$\boxed{\lim_{\|P\| \to 0} U(f,P)}$$

what does it mean ?

$\|P\| = \max \Delta x_i$

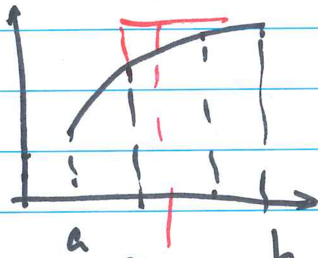
↳ norm of P

Lemma: With the notation as above:

Let P, Q be partitions on $[a,b]$. Then

(i) if $P \subseteq Q$, then we have

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$



(ii) $$\boxed{L(f,P) \leq U(f,Q)}$$

$\forall$ P, Q partitions

pf: (i):

Fact: We always have

$$(\star)\ \begin{cases} U(-f,Q) = -L(f,Q). \\ L(-f,Q) = -U(f,Q) \end{cases} \text{ easy!}$$

$\boxed{\circledast \text{ Claim: } U(f,Q) \leq U(f,P) \text{ if } P \subseteq Q}$

(Note that: If claim is O.K by $(\star)$

$\Rightarrow$ $L(f,P) \leq L(f,Q)$ (by considering $-f$)

③

Claim: $U(f,Q) \le U(f,P)$ if $P \subseteq Q$

pf: Put $l \equiv \#Q - \#P$

Using induction on l to obtain **Claim:**

As $l = 0 \Rightarrow$ claim is O.K ($\because P \subseteq Q \therefore P = Q$)

As $l = 1$.

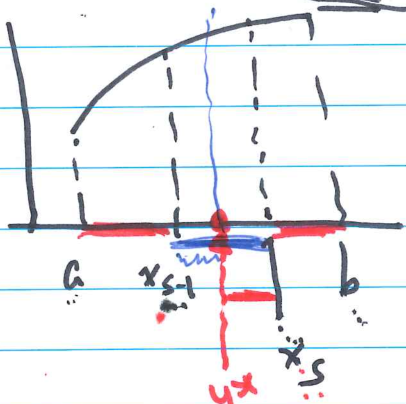
let $P = a = x_0 < x_1 < \cdots < x_n = b$

Since $P \subseteq Q$ and $\#Q - \#P = 1$,

$Q: a = x_0 < x_1 < \cdots < x_{s-1} < y^* < x_s < \cdots < x_n = b$ $\quad$ $1, \dots, n$ $\exists s$

for some $y^* \in [a,b]$, $y^* \in [x_{s-1}, x_s]$

Note that



$$M_s(f,P) = \sup\{f(x) \mid x \in [x_{s-1}, x_s]\}$$

$$\ge \begin{cases} \sup\{f(x) \mid x \in [x_{s-1}, y^*]\} \equiv \alpha. \\ \sup\{f(x) \mid x \in [y^*, x_s]\} \equiv \beta. \end{cases}$$

$\boxed{x_s - x_{s-1} = \Delta x_s}$

Then

$$U(f,P) - U(f,Q) = M_s(f,P)\,\Delta x_s - [\alpha(y^* - x_{s-1}) + \beta(x_s - y^*)]$$

$$= M_s(f,P)(y^* - x_{s-1}) + M_s(f,P)(x_s - y^*) - \alpha(y^* - x_{s-1}) - \beta(x_s - y^*)$$

$$= (M_s - \alpha)(y^* - x_{s-1}) + (M_s - \beta)(x_s - y^*) \ge 0 \qquad \square$$

(ii) Claim: $L(f,P) \le U(f,Q), \forall P, Q$

pf(ii) Let P, Q be partitions on [a,b]

Consider $\boxed{W = P \cup Q}$

Note W is a partition on $[a,b]$ and

$P \subseteq W$ and $Q \subseteq W$

by (i) $\Rightarrow$
$$L(f,P) \le L(f,W) \le U(f,W) \le U(f,Q)$$

$$\therefore \quad L(f,P) \le U(f,Q)$$

$\square$

---

* Def: Using the notation as above, write

(i) $\overline{\int_a^b} f(x)\,dx = \inf\{U(f,P) \mid P \text{ partition}\}$

(upper integral of f over $[a,b]$)

(ii) $\underline{\int_a}^b f(x)\,dx = \sup\{L(f,P) \mid P \text{ partition}\}$

(Lower integral of f)

Remark:

(i) Recall:

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a), \ \forall P$$

$\therefore$ $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$ always exist!

(ii) $\because$ $L(f,P) \le U(f,Q)$, $\forall P, Q$ partitions

if fix P, ~~runs~~ runs through Q

$$\overline{\int_a^b} f = \inf_Q U(f,Q)$$

$$\therefore \ L(f,P) \le \overline{\int_a^b} f, \ \forall P$$

$$\underline{\int_a^b} f = \sup_P L(f,P)$$

$$\Rightarrow \boxed{\underline{\int_a^b} f \le \overline{\int_a^b} f}$$

$(\star)$

e.g: Define $f:[0,1]\to\mathbb{R}$ by

$$f(x)=\begin{cases}1 & x\in\mathbb{Q}\cap[0,1]\\ 0 & x\in\mathbb{Q}^c\cap[0,1]\end{cases}$$

Note that if $P: 0=x_0<\cdots<x_n=1$, then

$$M_i(f,P)=1$$
$$m_i(f,P)=0 \quad , \quad \forall i$$

$$\therefore \quad U(f,P)=\sum_{i=1}^{n} M_i(f,P)\Delta x_i=1$$
$$L(f,P)=\sum_{i=1}^{n} m_i(f,P)\Delta x_i=0 \quad , \forall P$$

$$\therefore \quad \overline{\int_0^1} f(x)=\inf_P U(f,P)=1$$
$$\underline{\int_0^1} f(x)=\sup_P L(f,P)=0$$

$$\therefore \quad \underline{\int_0^1} f(x)\,dx \neq \overline{\int_0^1} f(x)\,dx$$

Def: Let $f : [a,b] \to \mathbb{R}$ be a bounded function

We say that f is Riemann integrable if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx \quad \left(\text{always have } \underline{\int_a^b} f \le \overline{\int_a^b} f\right)$$

Write $$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

Call: $\int_a^b f(x)\,dx$ the Riemann integral of f over $[a,b]$.

Denote $R[a,b]$ the class of Riemann integrable functions over $[a,b]$

Question:

What ~~is~~ does $R[a,b]$ look ~~look~~ like?

Expected:

$$C[a,b] \subseteq R[a,b] \;!$$

Prop: Using the notation as above, ~~we have~~ $\Rightarrow$

(i) $R[a,b]$ is a vector space (infinite dim vect sp)

(ii) $\int_a^b : R[a,b] \longrightarrow \mathbb{R}$:

$$\begin{aligned} &\ \ \downarrow \\ &f \longmapsto \int_a^b f(x)\,dx \end{aligned}$$ is a linear map

ie: (i) if $f, g \in R[a,b]$, $\alpha, \beta \in \mathbb{R}$ then

$$\alpha f + \beta g \in R[a,b]$$

(ii): $$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$$

pf: claim (1-1): If $f, g \in R[a,b] \Rightarrow f+g \in R[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

$\vdash_1'$: $$\boxed{\underline{\int_a^b} f + \underline{\int_a^b} g \le \underline{\int_a^b} (f+g)} \le \overline{\int_a^b} (f+g) \le \overline{\int_a^b} f + \overline{\int_a^b} g$$

∂(pf $\vdash_1'$): Note: we have

$$\left\{\begin{aligned} \overline{\int_a^b} f &= - \underline{\int_a^b} (-f) \\ \underline{\int_a^b} f &= - \overline{\int_a^b} (-f) \end{aligned}\right\} \leftarrow \text{by yourself} \quad \left(\because \begin{aligned} U(-f,P) &= -L(f,P) \\ L(-f,P) &= -U(f,P) \end{aligned} \right)$$

$\therefore$ it ~~needs~~ needs to show:

$$\underline{\int_a^b} f + \underline{\int_a^b} g \le \underline{\int_a^b} (f+g) \quad \cdots (\star)$$

(If $\star$ o.k. consider: $-f$, $-g$)

T_1'' : $\underline{\int_a^b} f + \underline{\int_a^b} g \le \underline{\int_a^b} (f+g)$?

pf T_1'' : Note we always have

$$\underline{L(f, P) + L(g, P) \le L(f+g, P)} \longleftarrow$$

(reason : $m_i(f, P) + m_i(g, P) \le m_i(f+g, P)$

i.e : $\inf\{f(x) \mid x \in [x_{i-1}, x_i]\} + \inf\{g(x) \mid x \in [x_{i-1}, x_i]\}$

$\le \inf\{f(x) + g(x) \mid x \in [x_{i-1}, x_i]\}$

$\therefore$ if, P_1, P_2 are partitions on $[a, b]$, $\Rightarrow$

$L(f, P_1) + L(g, P_2) \le$

$L(f, P_1 \cup P_2) + L(g, P_1 \cup P_2)$

$\le L(f+g, P_1 \cup P_2)$

$\le \sup\limits_{Q} L(f+g, Q) = \underline{\int_a^b}(f+g)$

$\forall P_1, P_2$)

Fix P_2,

$\Rightarrow \underline{\int_a^b} f + L(g, P_2) \le \underline{\int_a^b}(f+g)$

$$\boxed{\Rightarrow \underline{\int_a^b} f + \underline{\int_a^b} g \le \underline{\int_a^b}(f+g).}$$

//

(10)

$\therefore$ By considering $-f, -g$, we get

$$\overline{\int_a^b}(f+g) \le \overline{\int_a^b} f + \overline{\int_a^b} g$$

$\therefore$

$$\underline{\int_a^b} f + \underline{\int_a^b} g \le \underline{\int_a^b}(f+g) \le \overline{\int_a^b}(f+g) \le \overline{\int_a^b} f + \overline{\int_a^b} g \quad \cdots (*)$$

$\therefore$ if $f, g \in R[a,b] \overset{(*)}{\Rightarrow}$

$$\underline{\int_a^b}(f+g) = \overline{\int_a^b}(f+g)$$
$$= \overline{\int_a^b} f + \overline{\int_a^b} g$$
$$= \int_a^b f + \int_a^b g \qquad ///$$

F_2: If $\alpha \in \mathbb{R}$, $f \in R[a,b] \Rightarrow \alpha f \in R[a,b]$

and $\boxed{\int_a^b (\alpha f) = \alpha \int_a^b f}$

(pf F_2): Consider $\alpha \ge 0 \Rightarrow \begin{cases} \overline{\int_a^b} \alpha f = \alpha \overline{\int_a^b} f \;\checkmark \\ \underline{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f \;\checkmark \end{cases}$

As $\alpha < 0 \Rightarrow \begin{cases} \overline{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f \;\checkmark\checkmark \\ \underline{\int_a^b} \alpha f = \alpha \overline{\int_a^b} f \;\checkmark\checkmark\;\square \end{cases}$