THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2020)
Suggested Solution of Homework 8: Section 8.2: 6, 9, 12

6. Let $f_n(x) := 1/(1+x)^n$ for $x \in [0,1]$. Find the pointwise limit f of the sequence (f_n) on $[0,1]$. Does (f_n) converge uniformly to f on $[0,1]$? (3 marks)

   Solution. For $x = 0$, $\lim_{n\to\infty} f_n(0) = 1$. For $x \in (0,1]$, we have $1 + x > 1$ and hence, $\lim_{n\to\infty} f_n(x) = 0$. Therefore, the limit function f is

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0,1]. \end{cases}$$

   The sequence (f_n) does not converge uniformly to f on $[0,1]$, because f_n are continuous for all n and the limit function f is not continuous.

9. Let $f_n(x) := x^n/n$ for $x \in [0,1]$. Show that the sequence (f_n) of differentiable functions converges uniformly to a differentiable function f on $[0,1]$, and that the sequence (f_n') converges on $[0,1]$ to a function g, but that $g(1) \neq f'(1)$. (4 marks)

   Solution. For each $x \in [0,1]$, we have

$$|f_n(x)| = \left|\frac{x^n}{n}\right| \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

   Therefore, f_n converges uniformly to $f \equiv 0$ on $[0,1]$. On the other hand, it is easy to see that the pointwise limit of the sequence (f_n'), where $f_n'(x) = x^{n-1}$, is

$$g(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

   Therefore, we have $g(1) = 1 \neq 0 = f'(1)$.

12. Show that $\lim \int_1^2 e^{-nx^2}\, dx = 0$. (3 marks)

   Solution. We claim that the sequence (e^{-nx^2}) of functions converges uniformly to zero function on $[1,2]$. After showing this, we may apply **8.2.4 Theorem** on p.251 to conclude that $\lim \int_1^2 e^{-nx^2}\, dx = \int_1^2 \lim e^{-nx^2}\, dx = 0$.

   Note that on the interval $[1,2]$,

$$|e^{-nx^2}| \leq e^{-n} \to 0 \quad \text{as } n \to \infty.$$

   Therefore, the claim is shown.