THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2020)
Suggested Solution of Homework 2: Section 5.4: 2, 3; Section 6.1: 10

2. Show that the function $f(x) := 1/x^2$ is uniformly continuous on $A := [1, \infty)$, but that it is not uniformly continuous on $B := (0, \infty)$. (3 marks)

   Solution.

   Note that

   $$f(x) - f(y) = \frac{(y-x)(y+x)}{x^2y^2} = (y-x)(\frac{1}{x^2y} + \frac{1}{xy^2})$$

   If $x, y \in A$, then $\frac{1}{x^2y}$ and $\frac{1}{xy^2} \leq 1$. Therefore, $|f(x) - f(y)| \leq 2|x - y|$ on A. f is Lipschitz and hence uniformly continuous on A.

   On the other hand, let $x_n = \frac{1}{n}$. Then, (x_n) is a Cauchy sequence in B, but $f(x_n) = n^2$ is not a Cauchy sequence. Therefore, f is not uniformly continuous on B. (c.f. Theorem 5.4.7)

3. Use the Nonuniform Continuity Criterion 5.4.2 to show that the following functions are not uniformly continuous on the given sets. (2 marks each)

   (a) $f(x) := x^2$, $A := [0, \infty)$.

   (b) $g(x) := \sin(1/x)$, $B := (0, \infty)$.

   Solution.

   (a) Let $\epsilon_0 = 1$ and $\delta > 0$. Let $x_\delta = \frac{2}{\delta}$, $y_\delta = \frac{2}{\delta} + \frac{\delta}{2}$. Then, $|x_\delta - y_\delta| = \frac{\delta}{2} < \delta$ and $|f(x_\delta) - f(y_\delta)| = (\frac{\delta}{2})^2 + 2 > \epsilon_0$.

   (b) Let $\epsilon_0 = 1$ and $x_n = \dfrac{1}{2\pi n}$, $y_n = \dfrac{1}{2\pi n + \frac{\pi}{2}}$ for each $n \in \mathbb{N}$. Then, $\lim\limits_{n\to\infty} |x_n - y_n| = |0 - 0| = 0$, and $|g(x_n) - g(y_n)| = 1 \geq \epsilon_0$.

10. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x^2 \sin(1/x^2)$ for $x \neq 0$, and $g(0) := 0$. Show that g is differentiable for all $x \in \mathbb{R}$. Also show that the derivative g' is not bounded on the interval $[-1, 1]$. (3 marks)

    Solution.

    Note that $1/x^2$ is differentiable on $\mathbb{R} \setminus \{0\}$ and $\sin x$ is differentiable on $\mathbb{R}$. By chain rule, the composite function $\sin(1/x^2)$ is differentiable on $\mathbb{R} \setminus \{0\}$. Therefore, the function g is differentiable on $\mathbb{R} \setminus \{0\}$. At $x = 0$, note that

    $$\frac{g(x) - g(0)}{x - 0} = \frac{x^2 \sin(1/x^2) - 0}{x - 0} = x \sin(1/x^2)$$

    By sandwich theorem, $\lim\limits_{x\to 0} \dfrac{g(x) - g(0)}{x - 0} = 0$. Therefore, g is differentiable for all $x \in \mathbb{R}$.

Notice that for $x \neq 0$, by chain rule, we have $g'(x) = 2x\sin(1/x^2) - \frac{2}{x}\cos(1/x^2)$. Let $x_n = \frac{1}{\sqrt{2\pi n}}$ for each $n \in \mathbb{N}$. Then, (x_n) is a sequence in $[-1, 1]$. Moreover, $g'(x_n) = -2\sqrt{2\pi n} \to -\infty$ as $n \to \infty$. Therefore, g' is not bounded on the interval $[-1, 1]$.