**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH2060B Mathematical Analysis II (Spring 2020)**
**Suggested Solution of Homework 10: Section 9.4: 1(a),(e), 2, 3**

1. Discuss the convergence and the uniform convergence of the series $\sum f_n$, where $f_n(x)$ is given by: (a) $(x^2+n^2)^{-1}$, (e) $x^n/(x^n+1)$ $(x \geq 0)$ (5 marks)

**Solution.**

(a) For each $x \in \mathbb{R}$, note that

$$\frac{1}{x^2+n^2} \leq \frac{1}{n^2}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, by M-test (**9.4.6**), we see that $\sum f_n$ converges uniformly on $\mathbb{R}$.

(e) First, we notice that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ \frac{1}{2} & \text{if } x = 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Therefore, if $\sum f_n(x)$ converges, then $x \in [0,1)$. We will show that

(i) For each $a \in (0,1)$, $\sum f_n$ converges uniformly on $[0,a]$.
(ii) $\sum f_n$ does not converge uniformly on $[0,1)$.

Notice that if (i) is shown, one of the consequence is that $\sum f_n(x)$ converges on $[0,1)$, but by (ii), this convergence is not uniform. To see (i), for each $x \in [0,a]$,

$$f_n(x) \leq \frac{a^n}{0+1} = a^n.$$

Since $\displaystyle\sum_{n=1}^{\infty} a^n < \infty$ $(a<1)$, M-test tells us that $\sum f_n$ converges uniformly on $[0,a]$.

To see (ii), we will use the fact that if $\sum f_n$ converges uniformly on $[0,1)$, then $\displaystyle\lim_{n\to\infty} \|f_n\| = 0$, where

$$\|f_n\| := \sup\{|f_n(x)| : x \in [0,1)\}.$$

However, $f_n(1) = 1/2$ for each $n \in \mathbb{N}$ and also by the continuity of the function $f_n$, we would conclude that $\|f_n\| \geq 1/2$, which contradicts to the fact above.

To do that. Suppose $\sum f_n$ converges uniformly on $[0,1)$. By Cauchy criterion, for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ so that for every $n \geq N$ and $p \geq 1$, we have

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| < \epsilon \quad \text{for all } x \in [0,1).$$

We will show that the sequence $\{f_n\}$ fails the criterion. Let $\epsilon = 1/4$. For each $N \in \mathbb{N}$, we put $n = N$ and $p = 1$. At the point $x = (1/2)^{1/(n+1)} \in [0, 1)$, we have

$$|f_{n+1}(x)| = \frac{\frac{1}{2}}{\frac{1}{2} + 1} = \frac{1}{3} > \epsilon.$$

This completes the proof.

2. If $\sum a_n$ is an absolutely convergent series, then the series $\sum a_n \sin nx$ is absolutely and uniformly convergent. (2 marks)

**Solution.** For every $x \in \mathbb{R}$, the sequence $\{\sin nx\}$ is bounded. By Q7(a) in Homework 9, we see that $\sum a_n \sin nx$ converges absolutely for every $x \in \mathbb{R}$. To see that $\sum a_n \sin nx$ converges uniformly on $\mathbb{R}$, for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$|a_n \sin nx| \leq |a_n|.$$

Notice that $\sum |a_n| < \infty$ by assumption. So M-test implies that $\sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly on $\mathbb{R}$. Indeed, M-test implies both absolute convergence and uniform convergence.

3. Let $(c_n)$ be a decreasing sequence of positive numbers. If $\sum c_n \sin nx$ is uniformly convergent, then $\lim(nc_n) = 0$. (3 marks)

**Solution.** Let $f_n(x) = c_n \sin nx$ and let $\epsilon > 0$. Using Cauchy criterion, there is some $N \in \mathbb{N}$ so that for every $n \geq N$ and $p \geq 1$, we have

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| < \epsilon \quad \text{for all } x \in \mathbb{R}.$$

In particular, if we choose $p = n$ and $x = \pi/4n$, then for $(n+1) \leq k \leq 2n$,

$$f_k(x) = c_k \sin \frac{k\pi}{4n} \geq c_k \sin \frac{n\pi}{4n} = \frac{c_k}{\sqrt{2}} > 0.$$

Since $(c_n)$ is a decreasing sequence, it follows that

$$\frac{nc_{2n}}{\sqrt{2}} \leq \frac{c_{n+1} + c_{n+2} + \cdots + c_{2n}}{\sqrt{2}} < \epsilon \quad \text{for every } n \geq N.$$

Therefore, $\lim_{n\to\infty} (2n)c_{2n} = 0$. On the other hand,

$$(2n+1)c_{2n+1} \leq (2n+1)c_{2n} = 2nc_{2n} + c_{2n} \to 0 \quad \text{as } n \to \infty.$$

These show that $\lim(nc_n) = 0$, because both $\lim_{n\to\infty} (2n)c_{2n} = \lim_{n\to\infty} (2n+1)c_{2n+1} = 0$.