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17/02/2019

Recall : Mean Value th:

Let $f : [a,b] \to \mathbb{R}$ be a cont funct
Assume that f' exists on (a,b)
Then $\exists\, c \in (a,b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Prop: (Cauchy mean value th)

Assume that $f, g : [a,b] \to \mathbb{R}$. be cont functions.

Assume f', g' exists on (a,b). $g'(x) \neq 0$. $\forall x \in (a,b)$

$$\boxed{\text{Then } \exists\, c \in (a,b) \text{ s.t } \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}}$$

Note that put $g(x) \equiv x$, $\forall x \in (a,b)$

Pf: Define $\psi : [a,b] \longrightarrow \mathbb{R}$ ②

$$\psi(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)(g(x) - g(a))$$

Note: ψ is cont on $[a,b]$

- $\psi'(x)$ exists on (a,b)

$$\psi'(x) = f'(x) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right) g'(x).$$

- $\psi(a) = \psi(b) = 0$

Rolle's th $\Rightarrow \exists\, c \in (a,b)$

st $\psi'(c) = 0$

$$\therefore \quad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$\square$

L'Hospital th:

Let $f, g: (a,b) \to \mathbb{R}$ be diff functs on (a,b). Suppose that

$$\boxed{\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0} \quad \text{and} \quad (x_0 \in (a,b))$$

$g'(x) \neq 0, \ \forall x \in (a,b)$

If $\displaystyle\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l$ exists, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = l$$

pf: Note that want to show

$$\lim_{x \to x_0+} \frac{f(x)}{g(x)} = \lim_{x \to x_0-} \frac{f(x)}{g(x)} = l$$

$\vdash$: $\displaystyle\lim_{x \to x_0+} \frac{f(x)}{g(x)} = l$

Pf t: $\lim_{x \to x_0+} \frac{f(x)}{g(x)} = l$

Fix $x_0 < t < b$

by Cauchy mean value th: $\exists \; c(t) \in (x_0, t)$ s.t.

$$\frac{f(t)}{g(t)} = \frac{f(t) - f(x_0)}{g(t) - g(x_0)} = \frac{f'(c(t))}{g'(c(t))}$$

~~Since~~ $\because$ ~~c(t)~~ $x_0 < c(t) < t$,

$t \to x_0+ \Rightarrow c(t) \to x_0+$

$$\therefore \lim_{t \to x_0+} \frac{f(t)}{g(t)} = \lim_{t \to x_0+} \frac{f'(c(t))}{g'(c(t))} = l$$

$\square$

Polynomial functions

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$$a_i \in \mathbb{R}, \quad i = 0, 1, \ldots n, \quad x \in \mathbb{R}$$

Note: $f^{(k)}$ exists $\forall k = 0, 1, \ldots$

$f^{(k)} \equiv 0, \quad \forall k > n$

Notation:

$f^{(0)} \equiv f$

$f^{(1)} \equiv f'$

$\ldots$

$f^{(n)} \equiv f^{(n-1)}$

n- derivatives

$C^n(a,b) \equiv \{ f : (a,b) \to \mathbb{R} \mid f^{(k)}$ exists $k = 0, 1, \ldots n \}$

$C^\infty(a,b) \equiv \{ f : (a,b) \mid f^{(k)}$ exists $\forall k = 0, 1, 2 \ldots \}$

Smooth functions

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Prop: Let $f:(a,b)\to\mathbb{R}$ be C^{n} functions

i.e: $f^{(k)}$ exists, $\forall k=0,1,\dots n$.

Let $x_0\in(a,b)$ $\boxed{(\ast)\text{ Assume that } f^{(k)}(x_0)=0 \quad \forall k=0,\dots n-1}$

Then for each $x_0<x$,

$\exists z\in(x_0,x)$ s.t.

$$\boxed{f(x)=\frac{f^{(n)}(z)}{n!}(x-x_0)^n}$$

pf: Fix $x_0<x<b$

Put $\boxed{g(t)=(t-x_0)^{n}}$, $t\in[x_0,x]$

Then $g'(t)=n(t-x_0)^{n-1}$

by Cauchy mean value th

$\exists x_1\in(x_0,x)$ st

$$\frac{f(x)}{g(x)}=\frac{f(x)-\overset{0}{\overset{\shortparallel}{f(x_0)}}}{g(x)-\underset{0}{\underset{\shortparallel}{g(x_0)}}}=\frac{f'(x_1)}{g'(x_1)}$$

Consider: $[x_0,x_1]$

$f,g\big|_{[x_0,x_1]}$

Cauchy mean Value th.

$\Rightarrow \exists x_2 \in (x_0, x_1)$ s-t.

$$\frac{f'(x_1)}{g'(x_1)} = \frac{f'(x_1) - \overset{=0}{f'(x_0)}}{g'(x_1) - \underset{\overset{\shortparallel}{0}}{g'(x_0)}} = \frac{f''(x_2)}{g''(x_2)}$$

To repeat the same step,

$\therefore \exists x_n \in (x_0, x_{n-1})$ s-t

$$\frac{f^{(n-1)}(x_{n-1})}{g^{(n-1)}(x_{n-1})} = \frac{f^{(n-1)}(x_{n-1}) - \overset{=0}{f^{(n-1)}(x_0)}}{g^{(n-1)}(x_{n-1}) - \underset{\overset{\shortparallel}{0}}{g^{(n-1)}(x_0)}} = \frac{f^{(n)}(x_n)}{g^{(n)}(x_n)}$$

$$\therefore \frac{f(x)}{g(x)} = \frac{f'(x_1)}{g'(x_1)} = \frac{f^{(2)}(x_2)}{g''(x_2)} = \cdots = \frac{f^{(n)}(x_n)}{g^{(n)}(x_n)}$$

$$\boxed{\text{Note} \quad g^{(n)}(x_n) = n!}$$

$$\therefore \boxed{f(x) = \frac{f^{(n)}(x_n)}{n!}(x - x_0)^n}$$

$\square$

Th (Lagrange Remainder th):

Let $f \in C^n(a,b)$. Let $x_0 \in (a,b)$

Then $\forall\ x \in (a,b)$ (W.L.O.G: $x_0 < x < b$)

$\exists\ c \in (x_0, x)$ ($c \equiv c(x)$) s.t.

$$\boxed{f(x) = \underbrace{\sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{P(x)} + \frac{f^{(n)}(c)}{n!}(x-x_0)^n} \quad (\star)$$

Lagrange Remainder

(Recall the mean value th.

If $f \in C^1$ ~~then~~, $x_0 \in (a,b)$

$\exists\ c \in (x_0, x)$ s.t

$$f(x) - f(x_0) = f'(c)(x - x_0)$$

ie: $f(x) = f(x_0) + f'(c)(x - x_0)$

$\Rightarrow$ $\boxed{n=1}$ case $(\star)$.

Pf: put

$$p(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

put

$$E(x) \equiv f(x) - p(x)$$

Note:

$$E(x_0) = E'(x_0) = \cdots = E^{(n-1)}(x_0) = 0$$

$\therefore \exists\ c \in (x_0, x)$ s-t

$$E(x) = \frac{E^{(n)}(c)}{n!} (x-x_0)^n$$

$$\therefore \boxed{f(x) = p(x) + \frac{E^{(n)}(c)}{n!} (x-x_0)^n}$$

$\square$

e.g:

claim: $e \equiv 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots$ is irrational

ie: $e \neq \frac{p}{q}$, $p, q \in \mathbb{N}$, $q \neq 0$

pf: Consider $f(x)=e^x$, $x \in \mathbb{R}$,

N.B: $f \in C^\infty$ $0 \le x \le 1$

$f^{(k)}(x) = e^x$, $\forall k = 0, 1, 2 \ldots$

N.B:
Lagrange $\Rightarrow$ $\forall x$, $\forall n = 1, 2 \ldots$,

$\exists\, c \in (0, x)$ s.t

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}$$

$$\therefore\ 0 < \boxed{\frac{e^c}{(n+1)!}x^{n+1}} = e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) < \frac{3}{(n+1)!}$$

$\boxed{\forall n}$

$(\because 0 < x \le 1)$

$\boxed{0 < c < x \le 1}$

As $x = 1$

$$\boxed{0 < e - \left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) < \frac{3}{(n+1)!}}$$

$\boxed{\forall n}$