THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Tutorial Note 4
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Dual space of $C[a, b]$

Hahn-Banach theorem for normed spaces. Let f be a bounded linear functional on a subspace Z of a normed space X. Then there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f to X and has the same norm,

$$\|\tilde{f}\|_X = \|f\|_Z.$$

The Hahn-Banach theorem has many applications. One of them is to find a general representation formulas for bounded linear functionals on $C[a, b]$

The space $C[a, b]$ consists of continuous real-valued functions defined on the closed interval $[a, b]$. It is a vector space under pointwise addition and scalar multiplication and is infinite dimensional since $x^n \in C[a, b]$ for every $n \in \mathbb{N}$. The uniform norm is defined on $C[a, b]$ as

$$\|f\| = \sup_{0 \le x \le 1} |f(x)|, \quad f \in C[0, 1].$$

$C[0, 1]$ is complete under the metric $d(f, g) = \|f - g\|$. That is, if $f_n \to f$ then $f \in C[0, 1]$.

It turns out to consider elements of the dual space $(C[0, 1])^*$ which consists of the bounded linear functionals. A linear functional $l : C[0, 1] \to \mathbb{R}$ is in this space if and only if

$$f_n \to 0 \Longrightarrow l(f_n) \to 0.$$

Example 1. Fix $x_0 \in [0, 1]$ and define the Dirac mass at x_0,

$$\delta_{x_0}(f) = f(x_0).$$

This is clearly in the dual space and $\|\delta_{x_0}\| = 1$.

Example 2. Given a sequence of points $x_i \in [0, 1], i \in \mathbb{N}$ along with absolutely summable weights a_i, define

$$l(f) = \sum_i a_i f(x_i).$$

This is linear and we have $\|l\| \le \sum\limits_i |a_i|$ so it is in the dual space. In fact, $\|l\| = \sum_i |a_i|$ as can be seen by considering f^n with $f^n(x_i) = \operatorname{sgn}(a_i), i = 1, \cdots, n$.

Example 3. The Riemann integral is in the dual space. That is, the mapping

$$f \mapsto I(f) = \int_0^1 f \, dx$$

is linear and we have $\|I\| \le 1$ by the triangle inequality for integration

$$\left| \int f \, dx \right| \le \int |f| \, dx.$$

By choosing $f \equiv 1$ we can see $\|I\| = 1$.

The next example is more complicated and involves defining a different type of integral known as the Lebesgue-Stieljies integral.

Example 4. Lebesgue-Stieljies integration. A function w defined on $[0,1]$ is said to be of bounded variation if its total variation $\mathrm{Var}(w)$ on $[0,1]$ is finite, where

$$\mathrm{Var}(w) = \sup \sum_{k=1}^{n} |w(x_k) - w(x_{k-1})|,$$

the supremum being taken over all the partitions.

All functions of bounded variation on $[0,1]$ form a vector space. A norm on this space is given by

$$\|w\| = |w(0)| + \mathrm{Var}(w).$$

The normed space thus defined is denoted by $BV[0,1]$.

Given $w \in BV[0,1]$ with $w(0) = 0$ we define an integral via the flowing recipe. Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be a partition of $[0,1]$ and make the approximating sum

$$S(P, f, w) = \sum_{k=0}^{n-1} f(x_{k+1})(w(x_{k+1}) - w(x_k)).$$

If $f \in C[0,1]$ then as we refine the partition and take the mesh size $\delta \downarrow 0$, the sum converges to a number

$$\int_0^1 f\, dw = \lim_{\delta \downarrow 0} \sum_{k=0}^{n-1} f(x_{k+1})(w(x_{k+1}) - w(x_k)),$$

which is called the Riemann-Stieljies integral of f over $[0,1]$ with respect to w. Note that for $w(x) = x$, the integral is the familiar Riemann integral of f over $[0,1]$.

Also, if w has a derivative which is integrable on $[0,1]$, then

$$\int_0^1 f(x)\, dw(x) = \int_0^1 f(x) w'(x)\, dx.$$

Now from our definition it is clear that the integral is linear over both f and w. Moreover, we have the inequality

$$\left| \int_0^1 f\, dw \right| \leq \int_0^1 |f|\, d|w| \leq \mathrm{Var}(w) \|f\|$$

so that $\|I\| \leq \mathrm{Var}(w)$ and $I \in C[0,1]^*$. The representation theorem for bounded linear functionals on $C[0,1]$ by Riesz can be stated as follows.

Riesz's theorem for functionals on $C[0,1]$.

Given $l \in C[0,1]^*$ there exists $w \in BV, w(0) = 0$ so that

$$l(f) = \int_0^1 f\, dw, \quad \forall f \in C[0,1]. \tag{1}$$

And w has the total variation

$$\mathrm{Var}(w) = \|l\|.$$

Proof. From the Hahn-Banach theorem, l has an extension $\tilde{l}$ from $C[0,1]$ to the normed space $B[0,1]$ consisting of all bounded functions on $[0,1]$, together with

$$\|\tilde{l}\| = \|l\|.$$

We define the function w needed. For this purpose we consider the function f_x defined on $[0,1]$ by

$$f_x = \mathbf{1}_{[0,x]} \in B[0,1].$$

Using f_x and $\tilde{l}$, we define w on $[0,1]$ by

$$w(0) = 0, \quad w(x) = \tilde{l}(f_x), \quad x \in [0,1].$$

Claim: w is of bounded variation and $\mathrm{Var}(w) \leq \|l\|$.

Proof to the claim. For a complex number z, setting $\theta = \arg z$, we may write $z = |z|e(z)$ where

$$e(z) = \begin{cases} 1 & \text{if } \ z = 0 \\ e^{i\theta} & \text{if } \ z \neq 0 \end{cases}$$

Note that we have

$$|z| = z\overline{e(z)}.$$

For simplifying our formulas we write

$$\varepsilon_k = \overline{e(w(x_k) - w(x_{k-1}))}.$$

For any partition $0 = x_0 < x_1 < \cdots < x_n = 1$ we obtain

$$\begin{aligned}
\sum_{k=1}^{n} |w(x_k) - w(x_{k-1})| &= |\tilde{l}(f_{x_1})| + \sum_{k=2}^{n} |\tilde{l}(f_{x_k}) - \tilde{l}(f_{x_{k-1}})| \\
&= \varepsilon_1 \tilde{l}(f_{x_1}) + \sum_{k=2}^{n} \varepsilon_k \left[\tilde{l}(f_{x_k}) - \tilde{l}(f_{x_{k-1}})\right] \\
&= \tilde{l}\left[\varepsilon_1 f_{x_1} + \sum_{k=2}^{n} \varepsilon_k (f_{x_k} - f_{x_{k-1}})\right] \\
&\leq \|\tilde{l}\| \left\| \varepsilon_1 f_{x_1} + \sum_{k=2}^{n} \varepsilon_k (f_{x_k} - f_{x_{k-1}}) \right\|.
\end{aligned}$$

On the right, $\|\tilde{l}\| = \|l\|$ and the other factor equals 1 because $|\varepsilon_k|$ and from the definition of the f_{x_k}'s we see that for each $x \in [0,1]$ only one of the terms $f_{x_1}, f_{x_2} - f_{x_1}, \cdots$ is non-zero (and its norm is 1). On the left we take the supremum over all partitions of $[0,1]$. Then we have

$$\mathrm{Var}(w) \leq \|l\|. \tag{2}$$

Hence w is of bounded variation.

Proof of (1). For every partition P_n we define a function z_n by

$$z_n = f(x_0)f_{x_1} + \sum_{k=2}^{n} f(x_{k-1})(f_{x_k} - f_{k-1}). \tag{3}$$

Then $z_n \in B[0,1]$. By the definition of w, we have

$$\begin{aligned}
\tilde{l}(z_n) &= f(x_0)\tilde{l}(f_{x_1}) + \sum_{k=2}^{n} f(x_{k-1})\left[\tilde{l}(f_{x_k}) - \tilde{l}(f_{x_{k-1}})\right] \\
&= f(x_0)w(x_1) + \sum_{k=2}^{n} f(x_{k-1})\left[w(x_k) - w(x_{k-1})\right] \\
&= \sum_{k=1}^{n} f(x_{k-1})\left[w(x_k) - w(x_{k-1})\right],
\end{aligned}$$

where the last equality follows from $w(x_0) = w(0) = 0$.

Now we choose any sequence (P_n) of partitions of $[0,1]$ such that $\delta(P_n) \to 0$. As $n \to \infty$, the sum approaches the integral in (1). And hence it suffices to show that $\tilde{l}(z_n) \to \tilde{l}(f) = l(f)$ since $f \in C[0,1]$.

From the definition of f_x, we see that $z_n(0) = f(0)\cdot 1$. Furthermore, by (3), if $x_{k-1} < x \le x_k$, then we get $z_n(x) = f(x_{k-1}) \cdot 1$. It follows that for those x,

$$|z_n(x) - f(x)| = |f(x_{k-1}) - f(x)|.$$

Consequently, if $\delta(P_n) \to 0$, then $\|z_n - f\| \to 0$ because f in continuous and uniformly continuous on $[0,1]$. The continuity of $\tilde{l}$ implies that $\tilde{l}(z_n) \to \tilde{l}(f)$, so that (1) is established.

Finally, from (1) we have

$$|l(f)| \le \max|f(x)|\mathrm{Var}(w) = \|f\|\mathrm{Var}(w).$$

Taking the supremum over all $f \in C[0,1]$ with $\|f\| = 1$, we obtain $\|l\| \le \mathrm{Var}(w)$. Together with (2) we conclude that $\|l\| = \mathrm{Var}(w)$.