THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Tutorial Note 1
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Elementary inequalities

Jensen's inequality

(Finite form) For a real **convex function** φ, numbers $x_1, x_2, \cdots, x_n$ in its domain, and positive weights a_i, Jensen's inequality can be stated as

$$\varphi\left(\frac{1}{\sum a_i}\sum a_i x_i\right) \leq \frac{1}{\sum a_i}\sum a_i \varphi(x_i).$$

(Measure-theoretic and probabilistic form) Let (Ω, A, μ) be a probability space such that $\mu(\Omega) = 1$. If g is a real valued function which is μ-integrable and φ is a convex function on the real line, then

$$\varphi\left(\int_\Omega g\,d\mu\right) \leq \int_\Omega \varphi \circ g\,d\mu.$$

Young's inequality

(for products) In standard form, the inequality states that if a, b are nonnegative real numbers and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The equality holds if and only if $a^p = b^q$.

Proof: Consider a real-valued continuous and strictly increasing function f on $[0, c]$ with $c > 0$ and $f(0) = 0$. Let f^{-1} be its inverse function. Then for all $a \in [0, c]$ and $b \in [0, f(c)]$,

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx$$

with equality if and only if $b = f(a)$.

Put $f(x) = x^{p-1}$ and $f^{-1}(y) = y^{q-1}$ and it reduces to the required inequality.

The numbers p, q are said to be **Hölder conjugates** of each other.

Hölder's inequality

(for the counting measure) If p, q are **Hölder conjugates**, then

$$\sum |x_i y_i| \leq \left(\sum |x_i|^p\right)^{\frac{1}{p}} \left(\sum |y_i|^q\right)^{\frac{1}{q}}$$

for complex numbers $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$.

Proof: WLOG, we assume that $\sum |x_i|^p > 0, \sum |y_i|^q > 0$. Put

$$a = \frac{|x_i|}{(\sum |x_i|^p)^{\frac{1}{p}}}, \quad b = \frac{|y_i|}{(\sum |y_i|^q)^{\frac{1}{q}}}$$

in Young's inequality and we get

$$\frac{|x_i y_i|}{(\sum |x_i|^p)^{\frac{1}{p}} (\sum |y_i|^q)^{\frac{1}{q}}} \le \frac{|x_i|^p}{p \sum |x_i|^p} + \frac{|y_i|^q}{q \sum |y_i|^q}, \quad 1 \le i \le n.$$

Summing up over i,

$$\frac{\sum |x_i y_i|}{(\sum |x_i|^p)^{\frac{1}{p}} (\sum |y_i|^q)^{\frac{1}{q}}} \le \frac{\sum |x_i|^p}{p \sum |x_i|^p} + \frac{\sum |y_i|^q}{q \sum |y_i|^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

(for the Lebesgue measure) If Ω is a measurable subset of $\mathbb{R}^n$ with the Lebesgue measure and f, g are measurable complex-valued functions on Ω, then

$$\int_\Omega |f(x)g(x)|\, dx \le \left(\int_\Omega |f(x)|^p\, dx \right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q\, dx \right)^{\frac{1}{q}}.$$

Minkovski inequality

(for the counting measure) For any $p \ge 1$,

$$\left(\sum |x_i + y_i|^p \right)^{\frac{1}{p}} \le \left(\sum |x_i|^p \right)^{\frac{1}{p}} + \left(\sum |y_i|^p \right)^{\frac{1}{p}}$$

for complex numbers $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$.

Proof (of the case $p > 1$): From **Hölder inequality**,

$$\begin{aligned} \sum |x_i + y_i|^p &= \sum |x_i + y_i| |x_i + y_i|^{p-1} \le \sum |x_i| |x_i + y_i|^{p-1} + \sum |y_i| |x_i + y_i|^{p-1} \\ &\le \left(\sum |x_i|^p \right)^{\frac{1}{p}} \left(\sum |x_i + y_i|^{(p-1)q} \right)^{\frac{1}{q}} + \left(\sum |y_i|^p \right)^{\frac{1}{p}} \left(\sum |x_i + y_i|^{(p-1)q} \right)^{\frac{1}{q}} \\ &= \left(\sum |x_i|^p \right)^{\frac{1}{p}} \left(\sum |x_i + y_i|^p \right)^{\frac{1}{q}} + \left(\sum |y_i|^p \right)^{\frac{1}{p}} \left(\sum |x_i + y_i|^p \right)^{\frac{1}{q}}. \end{aligned}$$

Divide both sides by $\left(\sum |x_i + y_i|^p \right)^{\frac{1}{q}}$ and the desired inequality follows.

(for the Lebesgue measure) If Ω is a measurable subset of $\mathbb{R}^n$ with the Lebesgue measure and f, g are measurable complex-valued functions on Ω, then

$$\left(\int_\Omega |f(x) + g(x)|^p\, dx \right)^{\frac{1}{p}} \le \left(\int_\Omega |f(x)|^p\, dx \right)^{\frac{1}{p}} + \left(\int_\Omega |g(x)|^p\, dx \right)^{\frac{1}{p}}.$$

Banach spaces

Example: Let $1 \le p < \infty$. The space l^p is a Banach space.

Proof: 1. l^p is a normed space (omitted, triangle inequality comes from the Minkovski's inequality).

2. To show the completeness, we consider a Cauchy sequence $\{x_n\}$ in l^p, where $x_m = (x_1^m, x_2^m, \cdots)$. Then $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\forall m, n > N$,

$$\|x_m - x_n\| = \left(\sum_{j=1}^{\infty} \left| x_j^m - x_j^n \right|^p \right)^{\frac{1}{p}} < \varepsilon. \tag{1}$$

It follows that for every $j = 1, 2, \cdots$ we have

$$\left| x_j^m - x_j^n \right| < \varepsilon, \quad m, n > N.$$

We choose a fixed j. Then $(x_j^1, x_j^2, \cdots)$ is a Cauchy sequence of numbers and convergent, say, $x_j^m \to x_j$ as $m \to \infty$. Using these limits, we define $x = (x_1, x_2, \cdots)$ and desire to show that $x \in l^p$ and $x_m \to x$.

From (1) we have $\forall m, n > N$,

$$\sum_{j=1}^{k} \left| x_j^m - x_j^n \right|^p < \varepsilon^p, \quad k = 1, 2, \cdots.$$

Letting $n \to \infty$, we obtain for $m > N$

$$\sum_{j=1}^{k} \left| x_j^m - x_j \right|^p \le \varepsilon^p, \quad k = 1, 2, \cdots.$$

We may now let $k \to \infty$. Then for $m > N$,

$$\sum_{j=1}^{\infty} \left| x_j^m - x_j \right|^p \le \varepsilon^p. \tag{2}$$

This shows that $x_m - x \in l^p$. Since $x_m \in l^p$, it follows by means of Minkovski's inequality that

$$x = x_m + (x - x_m) \in l^p.$$

Furthermore, (2) implies that $x_m \to x$ and thus l^p is complete.

Remark. l^∞ is also a Banach space.

For $p \in [1, +\infty]$, the spaces l^p are increasing in p: for $1 \le p < q \le +\infty$, one has $\|f\|_q \le \|f\|_p$.

Example. $X = C(K)$, where K is a compact subset in $\mathbb{R}^n$. Define

$$\|f\|_X = \sup_{x \in K} |f(x)|.$$

Then $(X, \|\cdot\|_X)$ is a Banach space.

Proof: 1. $(X, \|\cdot\|_X)$ is normed space.

2. Completeness. Suppose f_n is a Cauchy sequence in X. Then $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall m, k > N$ and $x \in K$, it holds that

$$\sup_{x \in K} |f_m(x) - f_k(x)| < \varepsilon.$$

Therefore, for each $x \in K$, $\{f_m(x)\}$ is Cauchy in $\mathbb{R}^n$ and there exists a function $f(x)$ such that $\{f_m(x)\}$ converges pointwisely to $f(x)$ in K. Since N is independent of x, we can take $k \to \infty$ and consequently f_m converges uniformly to f.

Hence $f \in C(K)$ by compactness of K, which ends the proof.

Example. Let $K = [0,1]$. Define a norm $\|f\|_1 := \displaystyle\int_0^1 |f(x)|\, dx$. Then $(C[0,1], \|\cdot\|_1)$ is not a Banach space.

Consider the sequence

$$\{f_n(x)\}_{n \geq 2} = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2}, \\ n(x - \frac{1}{2}), & \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n}, \\ 1, & \frac{1}{2} + \frac{1}{n} < x \leq 1. \end{cases}$$

It's a Cauchy sequence in $(C[0,1], \|\cdot\|_1)$ since

$$\|f_n - f_m\|_1 = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| \to 0 \quad \text{as} \quad n, m \to \infty.$$

Let

$$f(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2}, \\ 1, & \frac{1}{2} < x \leq 1. \end{cases}$$

Then

$$\|f_n - f\|_1 = \frac{1}{2n} \to 0 \quad \text{as} \quad n \to \infty.$$

However, $f \notin C[0,1]$ and hence $(C[0,1], \|\cdot\|_1)$ is not a Banach space.