MATH 2060B - HW 7 - Solutions[1]

1 (P.280 Q1c,d)**.** For each of the following series,

i. determine if it converges

ii. determine if it converges absolutely

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n+2}$ b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n)}{n}$

Solution.

a. i. The series does not converge. Write $x_n := \frac{(-1)^{n+1}n}{n+2}$ and $y_n := \frac{(-1)^{n+1}}{n+2}$. Then $x_n = (-1)^{n+1} - 2\frac{(-1)^{n+1}}{n+2} = (-1)^{n+1} - 2y_n$. Suppose it were true that $\sum_n x_n$ converges. From the convergence of alternating harmonic series, it follows that $\sum_n y_n$ converges. Hence by considering linear combination of series, the series $\sum_n x_n + 2y_n = \sum_n (-1)^{n+1}$ converges. However it is well know that $\sum_n (-1)^{n+1}$ does not converge. Contradiction aries.

ii. Since the series does not converge, it does not converge absolutely.

b. i. The series converge. Write $x_n := (-1)^{n+1}\ln(n)/n$ and $y_n := \ln n/n$. Then $x_n = (-1)^{n+1}y_n$ for all $n \in \mathbb{N}$. By considering the function $f(x) := \ln(x)/x$ with derivative $f'(x) = (1-\ln(x))/x^2$ on $(0,\infty)$. By the Mean Value Theorem, it follows that (y_n) is non-negative decreasing when $n \geq 3$. Furthermore, $\lim_{x\to\infty}\ln(x)/x = \lim_{x\to\infty} 1/x = 0$ by the L'Hospital Rule. It follows that $\lim_n y_n = 0$ by sequential criteria. Hence, by the alternating series test $\sum_{n\geq 3} x_n = \lim_{n\geq 3}(-1)^{n+1}y_n$ converges. It follows clearly that $\sum_{n\geq 1} x_n$ converges as well.

ii. The series does not converge absolutely. With the above notation, $|x_n| = y_n$ for all $n \geq 1$. Note that $y_n = f(n)$ for all $n \geq 1$. Since f is continuous, non-negative decreasing on $[3,\infty)$, it follows from the integral test that $\sum_{n\geq 3} y_n$ converges if and only if $\int_3^\infty f(x)dx$ exists. By the Fundamental Theorem of Calculus, for all $b \geq 3$, we have

$$\int_3^b f(x)dx = \int_3^b \frac{\ln(x)}{n}dx = \frac{1}{2}\left(\ln(x))^2\right]_3^b$$

which diverges to ∞ as $b \to \infty$ (why?). Hence, the improper integral does not exists and so $\sum_{n\geq 3} y_n$ does not converge and so as $\sum_{n\geq 1}|x_n| = \sum_{n\geq 1} y_n$.

Comment.

- An alternative (and easier) solution for Q1a(i) is to use the m-term test by showing that $\lim_n x_n \neq 0$ where $x_n := \frac{(-1)^{n+1}n}{n+2}$. This can be shown by for example considering subsequences like (x_{2n}) or the absolute valued sequence $(|x_n|)$.
- For Q1b(i), the alternating series test states that if $(|y_n|)$ is a non-negative decreasing sequence with $\lim y_n = 0$, then the series $\sum_n (-1)^{n+1}y_n$ converges. It can be proved along the same line of thought using the Summation by Part formula as in the solution of Q2 and 3.
- An alternative (and easier) solution for Q1b(ii) is to use the comparison test together with the divergence of the harmonic series and the fact that $\frac{\ln(n)}{n} \geq \frac{1}{n} \geq 0$ for $n \geq 3$.
- There is no differentiation for functions defined on the domain $\mathbb{N}$. Make sure to define a function on some open intervals of $\mathbb{R}$ to work with *and then* induce properties for the related sequence using results you have learnt.
- Make sure you have a continuous, non-negative and decreasing function on a suitable domain when you are using the integral test.

[1]Please feel free to email your TA at kllam@math.cuhk.edu.hk for any questions concerning homework.

Question 2 and 3 can be done with the help of the Abel's Summation By Part Formula.

Lemma 0.1 (Summation By Part)**.** *Let (x_n), (a_n) be sequences of real numbers and (s_n) be the sequence of partial sum of the series $\sum a_n$. Then it follows that for all natural numbers $n \geq 2$, we have*

$$\sum_{i=1}^{n} a_i x_i = s_n x_n - \sum_{i=1}^{n-1} s_i (x_{i+1} - x_i)$$

Proof. Let $n \geq 2$. Then

$$\sum_{i=2}^{n} x_i s_i - x_{i-1} s_{i-1} = x_n s_n - x_1 s_1 = x_n s_n - x_1 a_1$$

On the other hand,

$$\begin{aligned}\sum_{i=2}^{n} x_i s_i - x_{i-1} s_{i-1} &= \sum_{i=2}^{n} x_i s_i - x_i s_{i=1} + x_i s_{i-1} - x_{i-1} s_{i-1} \\ &= \sum_{i=2}^{n} x_i (s_i - s_{i-1}) + \sum_{i=2}^{n} (x_i - x_{i-1}) s_{i-1} \\ &= \sum_{i=2}^{n} x_i a_i + \sum_{i=2}^{n} (x_i - x_{i-1}) s_{i-1}\end{aligned}$$

By equating the above, it follows that

$$\sum_{i=2}^{n} x_i a_i + \sum_{i=2}^{n} (x_i - x_{i-1}) s_{i-1} = x_n s_n - x_1 a_1$$

$$\Longrightarrow \quad \sum_{i=1}^{n} x_i a_i = x_n s_n - \sum_{i=2}^{n} (x_i - x_{i-1}) s_{i-1}$$

$$\Longrightarrow \quad \sum_{i=1}^{n} x_i a_i = x_n s_n - \sum_{i-1}^{n-1} (x_{i+1} - x_i) s_i$$

$$\Longrightarrow \quad \sum_{i=1}^{n} a_i x_i = s_n x_n - \sum_{i=1}^{n-1} s_i (x_{i+1} - x_i)$$

□

Remark.

- In general, let $(x_n), (a_n)$ be sequences of real numbers and (s_n) be the sequence of partial sum of (a_n). Then for all $n > m \geq 1$, one can compute using the above that

$$\sum_{i=m}^{n} a_i x_i = s_n x_n - s_{m-1} x_m - \sum_{i=m}^{n-1} s_i (x_{i+1} - x_i)$$

- Let $\sum a_n$ be a series. By considering the terms $a_n = s_n - s_{n-1}$ as "derivatives" of the partial sum (s_n), one shall see the analog of the summation by part formula with the integration by part formula, which states that for f, g continuous differentiable on $\mathbb{R}$, we have

$$\int_a^b f(t) dg(t) = f(b)g(b) - f(a)g(a) - \int_a^b g(t) df(t)$$

  where $\int_a^b f(t) dg(t) := \int_a^b f(t) g'(t) dt$ and $\int_a^b g(t) f(t) := \int_a^b g(t) f'(t) dt$ for all $b \geq a$ where $a, b \in \mathbb{R}$.

2 (P.280 Q9)**.** Let $\sum a_n$ be a series. Suppose the sequence of partial sum (s_n) of the series $\sum a_n$ is bounded. Show that the series $\sum_{n=1}^{\infty} a_n e^{-nt}$ converges for $t > 0$.

Solution. Let $x_n := e^{-nt}$. Then by the summation by part formula, we have for $n \geq 2$ that

$$\sum_{i=1}^{n} a_n e^{-nt} = \sum_{i=1}^{n} a_n x_n = s_n x_n - \sum_{i=1}^{n-1} s_i(x_{i+1} - x_i)$$

It suffices to show that $(s_n x_n)$ and $(\sum_{i=1}^{n-1} s_i(x_{i+1} - x_i))$ converges.
First, since $|s_n x_n| = |s_n||x_n| \leq \|(s_n)\|_\infty |x_n|$ where $\|(s_n)\|_\infty := \sup_n |s_n| < \infty$ and it is clear that $\lim_n x_n = 0$, it follows from the sandwich theorem that $\lim_n s_n x_n$ exists.
Next, we claim that $\sum s_i(x_{i+1} - x_i)$ converges absolutely. Note that (x_n) is a non-negative decreasing function by considering the derivative of the function $f(x) := e^{-xt}$ on $(0, \infty)$. It follows that for all $i \in \mathbb{N}$, we have

$$|s_i(x_{i+1} - x_i)| = |s_i|(x_i - x_{i+1}) \leq \|(s_n)\|_\infty (x_i - x_{i+1})$$

where $\lim_n \sum_{i=1}^{n} x_i - x_{i+1} = \lim_n x_1 - x_{n+1} = x_1$. It follows from the comparison test $\sum s_i(x_{i+1} - x_i)$ converges absolutely. Combining the two convergence, we have that $\sum_{i=1}^{\infty} a_n e^{-nt}$ converges.

Comment.

- The Dirichlet Test says that if (x_n) is a decreasing sequence with $\lim_n x_n = 0$, and $\sum a_n$ with bounded partial sum (s_n), then the series $\sum a_n x_n$ converges. This test is an immediate solution to this question; in fact one can obtain its proof by slightly modifying the above solution.

3 (P.280 Q14)**.** Let $\sum_{k=1}^{\infty} a_k$ be a series with sequence of partial sums (s_n). Suppose there exists $r < 1$ and $M > 0$ such that $|s_n| \leq Mn^r$ for all $n \in \mathbb{N}$. Show that the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Solution. Let $x_n := 1/n$. Then by the summation by part forumla, we have for all $n \geq 2$

$$\sum_{i=1}^{n} a_i x_i = s_n x_n - \sum_{i=1}^{n-1} s_i(x_{i+1} - x_i)$$

It suffices to show that $(s_n x_n)$ and $(\sum_{i=1}^{n-1} s_i(x_{i+1} - x_i))$ converges.
First $|s_n x_n| \leq Mn^r n^{-1} = Mn^{r-1}$ for all $n \in \mathbb{N}$. Since $r - 1 < 0$, we have $\lim_n n^{r-1} = 0$. It follows from the sandwich theorem that $\lim_n s_n x_n = 0$ and so exists.
Next, we claim that $\sum s_i(x_{i+1} - x_i)$ converges absolutely. Note that since $(x_n := 1/n)$ is clearly non-negative decreasing, for all $n \in \mathbb{N}$, we have

$$|s_n(x_{n+1} - x_n)| = |s_n|(x_n - x_{n+1}) = |s_n| \frac{1}{n(n+1)} \leq Mn^r \frac{1}{n^2} = \frac{M}{n^{2-r}}$$

Since $r < 1$, we have $2 - r > 1$. Therefore the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2-r}}$ converges. It follows from the comparison test that $\sum s_i(x_{i+1} - x_i)$ converges absolutely and so converges.
Combining the two convergence, we have that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Comment.

- It is crucial to emphasize that $2 - r > 1$ and so the related p-series converges.