MATH 2060B - Home Test 3

Suggested Solutions *(It does not reflect the marking scheme)*

1. (15 points)

Let $f(x) := \sum_{n=1}^{\infty} x^n(1-x)$. Let $D := \{x \in \mathbb{R} : f(x) \text{ is convergent}\}$.

(a) Find D.

(b) Does $f(x)$ converge uniformly on D?

Solution.

(a) $D = (-1, 1]$. For each $n \in \mathbb{N}$, put

$$u_n(x) = x^n(1-x) \quad \text{and hence} \quad f(x) = \sum_{n=1}^{\infty} u_n(x).$$

Consider the following cases:

- If $x = 0$ or $x = 1$, then $u_n(x) = 0$ for all $n \in \mathbb{N}$. Hence $f(x)$ is convergent.
- If $0 < |x| < 1$ or $|x| > 1$, then

$$\lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}(1-x)}{x^n(1-x)} \right| = |x|.$$

  Hence the **Ratio Test** implies that $f(x)$ is (absolutely) convergent if $0 < |x| < 1$ and $f(x)$ is divergent if $|x| > 1$.
- If $x = -1$, then $u_n(x) = 2(-1)^n$, which does not converge to 0. The **n-th Term Test** implies that $f(x)$ is divergent.

Combining the above observations, we have $D = (-1, 1]$.

Remark. It is not enough to show that $f(x)$ is convergent for $x \in (-1, 1]$. This only implies that $(-1, 1] \subseteq D$. We should also show that $f(x)$ is divergent for $x \notin (-1, 1]$.

(b) f does not converge uniformly on D. We compute the pointwise limit of f on D:

- If $x \in (-1, 1)$, then

$$f(x) = \sum_{n=1}^{\infty} x^n(1-x) = (1-x) \cdot \sum_{n=1}^{\infty} x^n = (1-x) \cdot \frac{x}{1-x} = x.$$

- If $x = 1$, then $f(x) = 0$.

Notice that each u_n is continuous on D. If f converges uniformly on D, then f is also continuous on D. However,

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} x = 1 \neq 0 = f(1).$$

This is a contradiction.

Comment.

- For (a), many students write

$$f(x) = \sum_{n=1}^{\infty} x^n(1-x) = (1-x) \sum_{n=1}^{\infty} x^n$$

  and claim that $f(x)$ is convergent if and only if $\sum x^n$ is convergent. Notice that we can only pull out the factor $(1-x)$ when the infinite sum is convergent. This argument is not valid.

- For (a), be careful that the terms cannot be zero if we want to apply the Ratio Test. Many students did not aware that $u_n(0) = u_n(1) = 0$ and apply Ratio Test to these two special cases.

2. (15 points)

Let g be a real analytic function on $\mathbb{R}$.

(a) Suppose that there is $\delta > 0$ such that $g(x) = 0$ for all $x \in (-\delta, \delta)$. Show that $g \equiv 0$ on $\mathbb{R}$. (Hint: Consider the set $\{r > 0 : \ g \equiv 0 \text{ on } (-r, r)\}$.)

(b) Show that if $\int_a^b |g(x)|dx = 0$ for some $a < b$, then $g(x) \equiv 0$ on $\mathbb{R}$.

Solution.

(a) We prove by contradiction. Suppose it were true that $g \not\equiv 0$ on $\mathbb{R}$. Denote $E := \{r > 0 : g = 0 \text{ on } (-r, r)\}$. Then E is non-empty by the assumption. Furthermore, it is bounded since otherwise we would have $g \equiv 0$ on $\mathbb{R}$. By the Axiom of completeness, we have $R := \sup E < \infty$. In fact we also have $R \geq \delta > 0$.

Next, we claim that g is *locally constantly zero* at R, that is, there exists $r > 0$ such that $g(x) = 0$ for all $x \in (-r + R, R + r)$. There are two ways to proceed.

Method 1: Computing Taylor's coefficients

First, we claim that $R \in E$, that is $g \equiv 0$ on $(-R, R)$. Suppose not. There exists $\xi \in (-R, R)$ such that $g(\xi) \neq 0$. Note $|\xi| < R$. Then by definition of supremum, there exists $r \in E$ such that $|\xi| < r$. It follows the $g(\xi) = 0$ as $\xi \in (-r, r)$. Contradiction arises. Therefore $g \equiv 0$ on $(-R, R)$. Note that it then follows that $g^{(k)} \equiv 0$ on $(-R, R)$. Since g is analytic on $\mathbb{R}$, it is smooth on $\mathbb{R}$. It follows that $g^{(k)}$ are continuous on $\mathbb{R}$, in particular at R. Hence, it follows that $g^{(k)}(R) = 0$ for all $k \in \mathbb{N}$. Since g is analytic at R, there exists $\delta^+ > 0$ such that

$$g(x) = \sum_{n=0}^{\infty} a_n (x - R)^n$$

for all $x \in (-\delta^+ + R, \delta^+ R)$ where $a_n = f^{(n)}(R)/n!$. It follows from previous computations that $a_n = 0$ for all $n \in \mathbb{N}$. Therefore, $g(x) = 0$ for all $x \in (-\delta^+ + R, \delta^+ R)$. It follows by definition that g is locally constantly zero at R.

Method 2: Considering the order of zeros

Since g is analytic on $\mathbb{R}$, g is analytic at R. Hence there exists $r > 0$ and a real sequence (a_n) such that

$$g(x) = \sum_{n=0}^{\infty} a_n (x - R)^n$$

for all $x \in (-r + R, R + r)$. We proceed to claim that $a_n = 0$ for all $n \in \mathbb{N}$.

Suppose not. Then $\{j \in \mathbb{N} \cup \{0\} : a_j \neq 0\} \neq \phi$. We take $N := \min\{j \in \mathbb{N} : a_j \neq 0\}$ by the well-ordering principle. From the minimality of N, we have

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} a_n (x - R)^n = \sum_{n=N}^{\infty} a_n (x - R)^n = (x - R)^N \sum_{n=N}^{\infty} a_n (x - R)^{n-N} \\ &= (x - R)^N \sum_{n=0}^{\infty} a_{n+N} (x - R)^n \end{aligned}$$

for all $x \in (-r + R, R + r)$. Now write $h : (-r + R, R + r) \to \mathbb{R}$ by $h(x) := \sum_{n=0}^{\infty} a_{n+N} (x - R)^N$. Note that the series defining h converges since it is just a scalar multiple of that of g pointwise, except maybe at R at which $h(R) = a_N$ is clearly defined. It follows that h is well-defined. We then have the equality

$$g(x) = (x - R)^N h(x)$$

for all $x \in (-r + R, R + r)$. Since h is a power series at R, it follows that h is a smooth function and so continuous at R. Note $h(R) = a_N \neq 0$. It follows that there exists $\rho > 0$ such that $h \neq 0$ on $(-\rho + R, R + \rho)$ by continuity. Further $(x - R)^N \neq 0$ for all $x \in (-\rho + R, R + \rho) \backslash \{R\}$ clearly.

It follows that $g(x) \neq 0$ for all $x \in (-\rho + R, R + \rho)\backslash\{R\}$. However, the definition of R tells us that $g(x) = 0$ for all $x \in (-\eta, \eta)$ where $0 < \eta < R$. Contradiction arises.
Therefore, it must then be the case that $a_n = 0$ for all $n \in \mathbb{N}$. Hence $g(x) = \sum_{n=0}^{\infty} a_n(x-R)^n = 0$ for all $x \in (-r + R, R + r)$ for some $r > 0$, that is g is locally constantly zero at R.

Now we have shown that g is locally constantly zero at R (and so at $-R$ with similar proof). Finally, we proceed to the last step:
let $r_1, r_2 > 0$ be such that $g \equiv 0$ on $(-r_1 - R, r_1 - R)$ and $(-r_2 + R, r_2 + R)$ respectively. Take $r := \min\{r_1, r_2\}$. It follows clearly that $R + r \in E$, which contradicts R being the supremum of E. .
Therefore it must be the case that E is unbounded and $g(x) = 0$ for all $x \in \mathbb{R}$.

(b) Suppose $\int_a^b |g(x)|dx = 0$ for some $a < b$. Note that g is analytic on (a, b) and so is continuous on (a, b). It follows that $|g|$ is non-negative continuous on (a, b). By Homework 4, Question 2, it follows that $|g|(x) := |g(x)| = 0$ for all $x \in (a, b)$. Hence, $g(x) = 0$ for all $x \in (a, b)$. Now write $c := \frac{a+b}{2}$ to be the mid-point of (a, b) and $r := c - a = b - c > 0$ the radius. It follows that $(a, b) = (-r + c, r + c)$. Note that the translated function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := g(x + c)$ is analytic by considering power series expansion of g with a simple substitution (with details to be filled by readers). Furthermore, h vanishes $(= 0)$ on $(-r, r)$. Hence, by part (a), it follows that $h \equiv 0$ on $\mathbb{R}$, which implies clearly that $g \equiv 0$ on $\mathbb{R}$.

Comment.

- Part (a) is a kind of extension problem: you are asked to extend the vanishing of the function g in a neighborhood of the origin to the entire real line. In general, extensions are made in the "boundary" of the *original* domain. That is why we consider the supremum of the set in the hint. If we consider instead some points in the *interor* of the neighborhood instead like $x_0 \in (-\delta, \delta)$, there is usually no result (for if we consider analyticity at x_0, the Taylor series may still work only in $(-\delta, \delta)$ if the radius of convergence is small).
- In Part (a), Method 2 in fact shows that zeros of an analytic function either are isolated or gives locally constantly zero neighborhood, that is, if $g(x) = 0$ for some $x \in \mathbb{R}$ and g analytic on $\mathbb{R}$, then either $g \equiv 0$ on $(-r+x, r+x)$ for some $r > 0$, or $g \neq 0$ anywhere on $(-r+x, r+x)\backslash\{x\}$ for some $r > 0$. The same is true if $\mathbb{R}$ is replaced by $\mathbb{C}$ and is an extremely important result in complex analysis on analytic (or holomorphic) functions.
- The first part of Part (b) about integrals basically follows from Assignment 4, Question 2.

3. (20 points)

For each $a \in \mathbb{R}$, put

$$a^+ = \begin{cases} a, & \text{if } a > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad a^- = \begin{cases} -a, & \text{if } a < 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Suppose that the series $\sum a_n$ is conditionally convergent, that is, the series $\sum a_n$ is convergent but $\sum |a_n| = \infty$. Show that $\sum a_n^+ = \sum a_n^- = \infty$.

(b) Consider $a_n := \frac{(-1)^{n+1}}{n}$ for $n = 1, 2, \ldots$. Show that there is a bijection σ on $\mathbb{Z}+$ such that $\liminf s_n = 0$ and $\limsup s_n = 1$, where $s_n := \sum_{k=1}^n a_{\sigma(k)}$.

Solution.

(a) First, note that we have the following equalities: for all $a \in \mathbb{R}$,

$$a^+ + a^- = |a| \tag{1}$$

$$a^+ - a^- = a \tag{2}$$

Now suppose the contrary. Then either $\sum a_n^+ < \infty$ or $\sum a_n^- < \infty$. (Note that (a_n^+) and (a_n^-) are sequences of non-negative number. Therefore by the Bounded Monotone Convergence, the corresponding series either exists or diverges to ∞.) Without loss of generality, we suppose $\sum a_n^+ < \infty$. Since we have $a_n^- = a_n - a_n^+$ for all $n \in \mathbb{N}$, by linearity of convergent series, it follows that $\sum_n a_n^- = \sum_n a_n - \sum_n a_n^+$ exists. Hence, it follows that $\sum_n |a_n| = \sum_n a_n^+ + \sum_n a_n^-$ converges. However, we have $\sum_n |a_n| = \infty$ by assumption. Contradiction arises.

(b) First, by the alternating series test, $\sum a_n$ is convergent. On the other hand $\sum |a_n|$ diverges since it gives the harmonic series. It follows that $\sum_n a_n$ converges conditionally. Write $(x_n := a_{2n-1})$ the sequence of positive terms and $(y_n := a_{2n-1})$ the sequence of negative terms.
Since $\sum a_n$ converges conditionally, it follows from part (a) that $\sum x_n = \sum a_n^+ = \infty$ and $\sum y_n = -\sum a_n^- = -\infty$.
Next we proceed to construct the required rearrangement.
Take $s_1 := \min\{j \in \mathbb{N} : \sum_{k=1}^j x_k > 1\}$. This is well-defined because we have $\sum x_n = \infty$.
If $s_1 = 1$, then we have

$$x_{s_1} \geq x_{s_1} - 1 \geq 0$$

Otherwise, by the minimality of s_1, we have

$$\sum_{k=1}^{s_1} x_k > 1 \geq \sum_{k=1}^{s_1 - 1} x_k$$

In any case, we have

$$x_{s_1} \geq \sum_{k=1}^{s_1} x_k - 1 \geq 0$$

Then we take $t_1 := \min\{j \in \mathbb{N} : \sum_{k=1}^{s} {}_1 x_k + \sum_{k=1}^{j} y_j \leq 0\}$. This is again well-defined (from $\sum y_n = -\infty$) and similarly we have

$$0 \geq \sum_{k=1}^{s_1} x_k + \sum_{k=1}^{t_1} y_k \geq y_{t_1}$$

where the second inequality follows from the minimality of t_1.
Then we proceed to define $s_2 := \min\{j \in \mathbb{N} : \sum_{k=s_1+1}^{j} x_k + \sum_{k=1}^{s_1} x_k + \sum_{k=1}^{t_1} y_k > 1\}$ and

$t_2 := \min\{j \in \mathbb{N} : \sum_{k=t_1+1}^{j} y_k + \sum_{k=1}^{s_2} x_k + \sum_{k=1}^{t_1} y_k < 0\}$; by repeating the process in general we have two strictly increasing seqeunce (s_n) and (t_n) defined by

$$s_n := \min\{j \in \mathbb{N} : \sum_{k=s_{n-1}+1}^{j} x_k + \sum_{k=1}^{s_{n-1}} x_k + \sum_{k=1}^{t_{n-1}} y_k > 1\}$$

$$t_n := \min\{j \in \mathbb{N} : \sum_{k=t_{n-1}+1}^{j} x_k + \sum_{k=1}^{s_n} x_k + \sum_{k=1}^{t_{n-1}} y_k > 1\}$$

for all $n \geq 2$. It follows from the minimality that we have

$$x_{s_n} \geq \sum_{k=1}^{s_n} x_k + \sum_{k=1}^{t_{n-1}} y_k - 1 \geq 0 \tag{1}$$

and

$$0 \geq \sum_{k=1}^{s_n} x_k + \sum_{k=1}^{t_n} y_k \geq y_{t_n} \tag{2}$$

for all $n \geq 2$ y.
Next we define the a function $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ by

$$\sigma(i) := \begin{cases} 2((i - t_n - s_n) + t_n) - 1 & ; t_n + s_n + 1 \leq i \leq t_{n+1} + s_n \\ 2((i - t_n - s_n) + s_n) & ; t_{n+1} + s_n + 1 \leq i \leq t_{n+1} + s_{n+1} \end{cases}$$

for all $t_n + s_n + 1 \leq i \leq t_{n+1} + s_{n+1}$ for all $n \in \mathbb{N}$ where $t_0 = s_0 := 0$. This function is well-defined on the domain because we have (s_n) and (t_n) being strictly increasing; we leave it to the readers to show its bijectivity and so σ is really a permutation. Note that the permuted sequence is given by

$$a_{\sigma(i)} := \begin{cases} x_{(i-t_n-s_n)+t_n} & ; t_n + s_n + 1 \leq i \leq t_{n+1} + s_n \\ y_{(i-t_n-s_n)+s_n} & ; t_{n+1} + s_n + 1 \leq i \leq t_{n+1} + s_{n+1} \end{cases}$$

for all $t_n + s_n + 1 \leq i \leq t_{n+1} + s_{n+1}$ for all $n \in \mathbb{N}$, which consists of alternative *blocks* of positive and negative terms of the original sequence. Furthermore, if we sum from the beginning of the permuted sequence, when we finish summing up a block of positive terms, the sequence is *just over* 1 (as indicated from the minimality of (s_n)) and when we finish summing up a block of negative terms, the sequence is *just below* 0 (as indicated from the minimality of (t_n)).
Rewriting the previous inequalities (1) and (2) in terms of the permutation, we have

$$x_{s_n} \geq \sum_{k=1}^{M(n)} a_{\sigma(k)} - 1 \geq 0 \tag{1'}$$

where $a_{\sigma(M(n))} = x_{s_n}$ for all $n \geq 2$

$$0 \geq \sum_{k=1}^{m(n)} a_{\sigma(k)} \geq y_{t_n} \tag{2'}$$

where $a_{\sigma(m(n))} = y_{t_n}$ for all $n \geq 2$.

Lastly, we show that σ is the required permutation. Write $s_n := \sum_{k=1}^{n} a_{\sigma(k)}$. Note that from the construction that for all $n \in \mathbb{N}$ we have

$$\sup_{k\geq n} s_k = \sup_{\substack{k\geq n \\ a_{\sigma(k)}=x_{s_i} \text{ for some } i}} s_k \tag{3}$$

and

$$\inf_{k\geq n} s_k = \inf_{\substack{k\geq n \\ a_{\sigma(k)}=y_{t_i} \text{ for some } i}} s_k \tag{4}$$

that is, when dealing with the supremum, we only need to consider when the summand finish summing up a positive block; and when dealing the the infermum, we only need to consider when the summand finish summing up a negative block.
Furthermore, as $\sum_n a_n$ converges, we have $\lim a_n = 0$. It follows by considering subsequences that $\lim_n x_{s_n} = \lim_n y_{t_n} = 0$. It then follows from (1'), (2'), (3), (4) and the squeeze theorem that $\limsup_n s_n = 1$ and $\liminf s_n = 0$.

Comment.

- For part (b), despite the technicality of the above solution, the more crucial main idea is to have a permuted sequence consisting of alternative *blocks* of positive and negative terms of the original sequence such that the partial sum is just above 1 (resp. below 0) when we finish summing up a block of positive terms (resp. negative terms).
- Readers may read Chapter 3 of *Principle of Mathematical Analysis, 3rd edition, McGraw Hill* by Rudin Walter (1976) (or the so-called Baby Rudin) for a textbook proof of Question 3.