MATH 2060B - Home Test 1

Suggested Solutions*(It does not reflect the marking scheme)*

1. (10 points) Let $f(x) = sgn(\sin\frac{\pi}{x})$ for $x \neq 0$ and $f(0) = 0$, where sgn denotes the sign function. Show that f is Riemann integrable over $[-1, 1]$ and find $\int_{-1}^{1} f(x)dx$.

Solution. For any integer $n \geq 2$, let $0 < \delta_n < 1/2n(n-1)$. Then we have

$$-\frac{1}{k} + \delta_n < -\frac{1}{k+1} - \delta_n \quad \text{and} \quad \frac{1}{k+1} + \delta_n < \frac{1}{k} - \delta_n, \quad \forall k = 1, 2, ..., n-1.$$

Consider the partition P_n of $[-1, 1]$ given by

$$P_n = \left\{-1, -1+\delta_n, -\frac{1}{2} \pm \delta_n, ..., -\frac{1}{n} - \delta_n, -\frac{1}{n}, \frac{1}{n}, \frac{1}{n} + \delta_n, ..., \frac{1}{2} \pm \delta_n, 1-\delta_n, 1\right\}.$$

By observing the graph of $\sin(\pi/x)$, the infimum and supremum of f on each sub-interval with respect to P_n can be determined:

- If $x \in [-1/k + \delta_n, -1/(k+1) - \delta_n]$ for some $k = 1, 2, ..., n-1$, we have
$$-\frac{1}{k} < x < -\frac{1}{k+1} \quad \Longrightarrow \quad -(k+1)\pi < \frac{\pi}{x} < -k\pi.$$
Hence $f(x) = 1$ if k is even and $f(x) = -1$ if k is odd.
- Similarly if $x \in [1/(k+1) + \delta_n, 1/k - \delta_n]$ for some $k = 1, 2, ..., n-1$, we have
$$\frac{1}{k+1} < x < \frac{1}{k} \quad \Longrightarrow \quad k\pi < \frac{\pi}{x} < (k+1)\pi.$$
Hence $f(x) = 1$ if k is odd and $f(x) = -1$ if k is even.
- If x is in the remaining sub-intervals, we have the universal bound: $-1 \leq f(x) \leq 1$.

Notice that the terms in the lower sum and the upper sums with respect to sub-intervals of the first and second type cancel out. Hence the lower and upper sums of f with respect to P_n are given by the terms with respect to sub-intervals of the third type:

$$L(f, P_n) \geq (-1) \cdot \left[2\left(\delta_n + \sum_{k=2}^{n-1} 2\delta_n + \delta_n\right) + \frac{2}{n}\right] = -4(n-1)\delta_n - \frac{2}{n} > -\frac{4}{n}$$
$$U(f, P_n) \leq 1 \cdot \left[2\left(\delta_n + \sum_{k=2}^{n-1} 2\delta_n + \delta_n\right) + \frac{2}{n}\right] = 4(n-1)\delta_n + \frac{2}{n} < \frac{4}{n}$$

It follows that the lower and upper integrals of f satisfy:

$$-\frac{4}{n} < L(f, P_n) \leq \underline{\int_{-1}^{1}} f(x)dx \leq \overline{\int_{-1}^{1}} f(x)dx \leq U(f, P_n) < \frac{4}{n}, \quad \forall n \geq 2.$$

Since $n \geq 2$ is arbitrary, taking limit on both sides gives

$$0 \leq \underline{\int_{-1}^{1}} f(x)dx \leq \overline{\int_{-1}^{1}} f(x)dx \leq 0.$$

It follows that the upper and lower integrals of f are both equal to 0. i.e., f is Riemann integrable over $[-1, 1]$. Moreover, we have

$$\int_{-1}^{1} f(x)dx = 0.$$

Comment. Many students take partitions in the form

$$P_n = \left\{ \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, ..., \pm \frac{1}{n} \right\}.$$

However, they mistakenly claim that $f(x) = 1$ or $f(x) = -1$ on each sub-interval except that contain the point 0. In fact, $f(x) = 0$ at every $x = 1/k$. Hence the supremum and infimum on these sub-intervals are not equal. This observation suggest we replace every $1/k$ by $1/k \pm \delta_n$, where δ_n is sufficiently small.
On the other hand, some students claim that f is integrable over $[1/(k+1), 1/k]$ for all $k \in$ and thus integrable over $[0, 1]$. This is not true as there are infinitely many intervals. Instead, we have $f \in \mathcal{R}[\varepsilon, 1]$ for any $\varepsilon > 0$. Then follows by a suitable argument we can have $f \in \mathcal{R}[0, 1]$.

2. (20 points) Let f be a continuous real-valued function defined on $\mathbb{R}$.

(a) Suppose that there are constants c_0 and c_1 such that

$$\lim_{x\to 0} \frac{f(x) - c_0 - c_1 x}{x} = 0.$$

Show that $f'(0)$ exists.

(b) Suppose that f is a C^1-function and there are constants c_0, c_1 and c_2 such that

$$\lim_{x\to 0} \frac{f(x) - c_0 - c_1 x - c_2 x^2}{x^2} = 0.$$

Does it imply that the second derivative of f at 0 exist? Prove your assertion.

Solution.

(a) First, we show that $f(0) = c_0$. Note that for all $x \neq 0$, we have

$$f(x) = \frac{f(x) - c_0}{x} x + c_0 = \frac{f(x) - c_0 - c_1 x}{x} x + c_1 x + c_0$$

Hence, by continuity of f at 0, we have

$$\begin{aligned} f(0) = \lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{f(x) - c_0 - c_1 x}{x} x + c_1 x + c_0 \\ &= \lim_{x\to 0} \frac{f(x) - c_0 - c_1 x}{x} \lim_{x\to 0} x + c_1 \lim_{x\to 0} x + \lim_{x\to 0} c_0 \\ &= 0 \cdot 0 + c_1 \cdot 0 + c_0 = c_0 \end{aligned}$$

Next we proceed to show $f'(0) = c_1$ and so $f'(0)$ exists.
This follows since we have

$$\begin{aligned} f'(0) := \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0} \frac{f(x) - c_0}{x} \\ &= \lim_{x\to 0} \frac{f(x) - c_0 - c_1 x}{x} + c_1 \\ &= 0 + c_1 = c_1 \end{aligned}$$

(b) No. We proceed to give a counterexample. Take $c_0 = c_1 = c_2 := 0$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^3 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

First, note that $\lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0} x \sin(1/x) = 0$. Hence, the condition in the question is satisfied.
Next, we claim that $f \in C^1(\mathbb{R})$.
For $x \neq 0$, it is clear that $f'(x) = 3x^2 \sin(1/x) - x\cos(1/x)$ exists.
When $x = 0$, we have $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x^2 \sin(1/x) = 0$. Therefore $f'(0) = 0$ exists.
It remains to show that f' is continuous at $x = 0$ (as it is clearly continuous when $x \neq 0$). This follows from the Squeeze Theorem as by the triangle inequality we have the inequality

$$|f'(x)| = \left|3x^2 \sin(1/x) - x\cos(1/x)\right| \leq 3\left|x^2\right| + |x|$$

where $x \neq 0$.
Finally, we show that $f''(0)$ does not exist.
Note that for all $x \neq 0$, we have $\frac{f'(x) - f'(0)}{x - 0} = 3x\sin(1/x) - \cos(1/x)$. Since $\lim_{x\to 0} \cos(1/x)$ does not exists (see Assignment 3 Q1) but $\lim_{x\to 0} x\sin(1/x) = 0$, it follows that $f''(0)$ does not exist.

Comment. In Part (b), many of you has claimed the truth of the statement by using the L'Hospital Rule on

$$\lim_{x\to 0} \frac{f(x) - c_0 - c_1 x - c_2 x^2}{x^2} = 0$$

to obtain

$$\lim_{x\to 0} \frac{f'(x) - c_1 - 2c_2 x}{2x} = 0$$

which is not correct. In general, the converse of the L'Hospital Rule may not hold: under the condition of L'Hospital Rule, the existence of $\lim_{x\to a} \frac{f(x)}{g(x)}$ may not imply the existence of $\lim_{x\to a} \frac{f'(x)}{g'(x)}$. In fact such concepts appeared in Assignment 3 Q1; the counter-example here is simply a slight modification of the function in that question.

3. (20 points) Let $f:(0,1)\to\mathbb{R}$ be a function given by

$$f(x)=\begin{cases}\frac{1}{p} & \text{if } x=\frac{q}{p} \text{ and } p,q \text{ are relatively prime positive integers;}\\ 0 & \text{if } x \text{ is irrational.}\end{cases}$$

a) Describe the continuity of f. b) Describe the differentiability of f.

Justify your answer by using the definitions.

Solution.

(a) We claim that f is discontinuous on $\mathbb{Q}\cap(0,1)$ but is continuous on $(0,1)$.
Claim 1: f is discontinuous on $\mathbb{Q}$.
Let $x\in(0,1)$ be rational. By denseness of $\mathbb{Q}^c\cap(0,1)$ in $(0,1)$, pick a sequence of irrational numbers (x_n) in $(0,1)$ such that $\lim_n x_n=x$. Then $\lim_n f(x_n)=0$, but $f(x)=1/p$ for some $p\in\mathbb{N}$. Hence f is not continuous at x by sequential criteria.
Claim 2: f is continuous on $\mathbb{Q}^c$.
Let $x\in(0,1)$ be irrational. Define $E_p:=\{x\in(0,1): x=\frac{q}{p}, q,p\in\mathbb{N},(q,p)=1\}$ for all $p\in\mathbb{N}$. Then E_p is a finte set for all $p\in\mathbb{N}$ and $\mathbb{Q}\cap(0,1)=\bigsqcup_p E_p$.
Now let $1>\epsilon>0$ (why can you restrict the value of ϵ?). Let $N:=\max\{n\in\mathbb{N}: n\leq 1/\epsilon\}$ (whose existence follows from the well-ordering principle of natural numbers as the concerned set is non-empty).Since E_p is a finite set for all $p\in\mathbb{N}$, we can choose $\delta>0$ small enough (how?) such that $B_\delta(x)\cap(0,1)\cap\bigsqcup_{p=1}^N E_p=\phi$.
Now take $y\in B_\delta(x)$. Suppose $y\notin\mathbb{Q}$. Then $|f(y)-f(x)|=|0-0|=0$ since both $x,y\notin\mathbb{Q}$.
Suppose $y\in\mathbb{Q}$. Note that $(0,1)\cap\mathbb{Q}=\bigsqcup_{p=1}^\infty E_p$. By the choice of δ, we have $y\in E_p$ for some $p>N$. By the maximality of N, we then have $p>1/\epsilon$ and so $1/p<\epsilon$.
Hence, $|f(y)-f(x)|=|1/p-0|<\epsilon$. We have showed by definition that f is continuous on all irrational points.

(b) We claim that f is nowhere differentiable on $(0,1)$. Since differentiability implies continuity, it is clear that f is not differentiable on rational points. It remains to show that f is not differentiable on irrational points.
Let $x\in(0,1)$ be irrational. We proceed to prove by contraction.
Suppose it were true that $f'(x)$ exists. By denseness of $\mathbb{Q}^c\cap(0,1)$. Choose a sequence (x_n) of irrational numbers where $x_n\neq x$ such that $\lim_n x_n=x$. Then we have for all $n\in\mathbb{N}$ the difference quotient to be

$$\frac{f(x_n)-f(x)}{x_n-x}=0$$

Hence, $f'(x)=0$. It then suffices to show that $f'(x)$ cannot be 0.
Consider the sequence of prime numbers (p_n) where p_n denotes the nth prime numbers. For all $n\in\mathbb{N}$, define $\delta_n:=1/p_n$. It is clear that $\lim_n\delta_n=0$ (since (p_n) is a subsequence of natural numbers). Since $x<1$ and we have $\lim_n x+\delta_n=x<1$, without loss of generality by considering tail sequences, we can assume $x+\delta_n<1$ for all $n\in\mathbb{N}$. Note that for all $n\in\mathbb{N}$, since x is irrational, the open intervals $(xp_n,(x+\delta_n)p_n)$ is of length 1 with non-integral endpoints. Hence, there exists $q_n\in\mathbb{N}$ such that

$$0<xp_n<q_n<(x+\delta_n)p_n<p_n$$

It follows that

$$0<x<\frac{q_n}{p_n}<x+\delta_n<1$$

and so $\left|\frac{q_n}{p_n}-x\right|<\delta_n=\frac{1}{p_n}$.
Finally, write $x_n:=\frac{q_n}{p_n}$. By the above, we have $\lim_n x_n=x$ and that

$$\left|\frac{f(x_n)-f(x)}{x_n-x}\right|=\left|\frac{1/p_n}{q_n/p_n-x}\right|\geq\frac{1/p_n}{1/p_n}=1$$

Hence $f'(x)\geq 1$ which is a contradiction.

Comment. The function in the question is called the Thomae's Function and the solution to this question could be found on the Internet, for example on Wikipedia. However, many of those solutions, for Part (b) in particular, make use of non-trivial number-theoritic results related to Diophantine Approximation (approximating real numbers by rational numbers) like the Hurwitz's Theorem. We should remark that those are not necessary as could be seen from the proof, which uses only the existence of infinitely many prime numbers.
Of course, if you are found to cite those number-theoretic results without a sound proof, you would lose a portion of marks.