MATH2050A HW5 Solution

9. Let $\epsilon > 0$ be given. By definition of $\lim_{x\to\infty} xf(x) = L$, there exists some $K(\epsilon) \in \mathbb{R}$ s.t. for any $x > K$, $|xf(x) - L| < 1$. In other words, $|xf(x)| < |L| + 1$. Consider $x > \max\{K, (|L| + 1)/\epsilon\} > 0$, $|f(x)| < (|L| + 1)/|x| < \epsilon$, which implies that $\lim_{x\to\infty} f(x) = 0$.

7. Consider $\epsilon = f(c)/10 > 0$, by the definition of continuity of f at c, there exists $\delta > 0$ s.t. if $x \in V_\delta(c)$, $|f(x) - f(c)| < f(c)/10$. In particular, $f(x) > 9f(c)/10 > 0$ as desired.

8. Since f is continuous on $\mathbb{R}$, f is continuous at x. By the Sequential Criterion for Contunuity, $\lim(x_n) = x$ implies $f(x) = \lim(f(x_n)) = \lim(0) = 0$. Therefore $x \in S$.

12. Since rational numbers are dense in $\mathbb{R}$, every real numbers can be written as limit of a sequence of rational numbers. As in question 8, since $\mathbb{Q} \subset S$ and f is continuous, we have $S = \mathbb{R}$ and $f(x) = 0$ for all $x \in \mathbb{R}$.