MATH2050A HW4 Solution

6. WLOG, by replacing x_n with $-x_n$ if necessary, we may assume that (x_n) tends to $+\infty$. Let $L = \lim(x_n y_n)$. Let $\epsilon > 0$ be given. By definition, (x_n) tends to $+\infty$ implies $\exists\ N_1 \in \mathbb{N}$ s.t. $x_n > \max(2, \frac{2|L|}{\epsilon})$ whenever $n > N_1$. Also, $L = \lim(x_n y_n)$ implies $\exists\ N_2 \in \mathbb{N}$ s.t. $|x_n y_n - L| < \epsilon$ whenever $n > N_2$. Now, $|y_n| \le |y_n - \frac{L}{x_n}| + |\frac{L}{x_n}| = \frac{|x_n y_n - L|}{|x_n|} + |\frac{L}{x_n}| < \epsilon/2 + |L|/(2|L|/\epsilon) = \epsilon$ whenever $n > \max(N_1, N_2)$. Since ϵ is arbitrary, by definition of limit of sequence, $\lim(y_n) = 0$

4. Consider $x_n = \frac{1}{n\pi}$, then $\cos(1/x_n) = 1$ if n is even but $\cos(1/x_n) = -1$ if n is odd. Thus $\{\cos(1/x_n)\}$ is does not exist. So $\lim_{x\to 0}\cos(1/x)$ does not exist by sequential criterion for limits.

   Since $-x \le x\cos(1/x) \le x$, and $\lim_{x\to 0} -x = \lim_{x\to 0} x = 0$. By squeeze theorem, $\lim_{x\to 0}\cos(1/x) = 0$.

5. Let $\epsilon > 0$ be given. f is bounded on a neighborbood of c implies there exists $\delta_1 > 0$ and $M > 0$ s.t. $|f(x)| < M$ whenever $x \in (c-\delta_1, c+\delta_1)$. Since $\lim_{x\to c} g(x) = 0$, there exists $\delta_2 > 0$ s.t. $|g(x)| < \epsilon/M$ whenever $x \in (c-\delta_2, c+\delta_2)$. Take $\delta = \min(\delta_1, \delta_2)$, then if $x \in (c-\delta, c+\delta)$, $|fg(x)| < M|g(x)| < \epsilon$. By definition of limit of functions, $\lim_{x\to c} fg(x) = 0$.

14. Let $\epsilon > 0$ be given. Let $L = \lim_{x\to c} f$. There exists $\delta > 0$ s.t. $|f(x) - L| < \epsilon$ whenever $x \in (c-\delta, c+\delta)$. However $||f(x)| - |L|| \le |f(x) - L| < \epsilon$ whenever $x \in (c-\delta, c+\delta)$. By defintion of limit of functions, $\lim_{x\to c}|f|(x) = |L| = |\lim_{x\to c} f(x)|$.