MATH2050A HW2 Solution

5. (a) Let $\epsilon > 0$ be given. By Archimedean Property, we can take $K(\epsilon) \in \mathbb{N}$ s.t. $K(\epsilon) > 1/\epsilon$, we have

$$n < \epsilon n^2 < \epsilon(n^2+1) \quad \forall n > K(\epsilon).$$

Since $n^2 + 1 > 0$ and $\frac{n}{n^2+1} > 0$,

$$\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \epsilon.$$

By definition of the limit of the sequence, we get the desired limit.

(c) Let $\epsilon > 0$ be given. By Archimedean Property, we can take $K(\epsilon) \in \mathbb{N}$ s.t. $K(\epsilon) > 13/(4\epsilon)$, we have

$$6.5 < 2n\epsilon < (2n+5)\epsilon \quad \forall n > K(\epsilon).$$

Since $2n + 5 > 0$,

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \frac{6.5}{2n+5} < \epsilon.$$

By definition of the limit of the sequence, we get the desired limit.

11. Let $\epsilon > 0$ be given. By Archimedean Property, we can take $K(\epsilon) \in \mathbb{N}$ s.t. $K(\epsilon) > \max\{1, 1/(\epsilon)\}$, we have

$$1 < \epsilon n < \epsilon n(n+1).$$

Since $n(n+1) > 0$ and $\frac{1}{n(n+1)} > 0$,

$$\left|(\frac{1}{n} - \frac{1}{n+1}) - 0\right| = \left|\frac{1}{n(n+1)}\right| = \frac{1}{n(n+1)} < \epsilon.$$

By definition of the limit of the sequence, we get the desired limit.

9. Note that

$$y_n = (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Since

$$\frac{1}{2}\sqrt{1-\frac{1}{n+1}} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+1}} < \sqrt{n}y_n < \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n}} = \frac{1}{2},$$

and

$$\lim \frac{1}{2}\sqrt{1-\frac{1}{n+1}} = \frac{1}{2}$$

by using Example 3.1.6. (limit of $1/n$), Thm 3.1.9 (limit of m-tail), Thm 3.2.3 (limit under subtraction) and Thm 3.2.10 (limit under square root), we have

$$\lim \sqrt{n}y_n = \frac{1}{2}$$

by Squeeze Theorem.