MMAT5010 2021 Assignment 9

Q1. The inner product of $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{C}^2$ is given by

$$(x, y) := x_1\bar{y_1} + x_2\bar{y_2}$$

where $\bar{y_i}$ denoted the complex conjugate of $y_i \in \mathbb{C}$. We compute $\langle Tx, y\rangle$:

$$\begin{aligned}\langle Tx, y\rangle &= \frac{1}{\sqrt{2}}(x_1\bar{y_1} + ix_2\bar{y_1} + x_1\bar{y_2} - ix_2\bar{y_2}) \\ &= x_1(\frac{1}{\sqrt{2}}(\overline{y_1 + y_2})) + x_2(\frac{1}{\sqrt{2}}(i\bar{y_1} - i\bar{y_2}))\end{aligned}$$

By definition, $\langle Tx, y\rangle = \langle x, T^*y\rangle = \overline{\langle T^*y, x\rangle}$, we have

$$\begin{aligned}\langle T^*y, x\rangle &= \bar{x_1}(\frac{1}{\sqrt{2}}(y_1 + y_2)) + \bar{x_2}(\frac{1}{\sqrt{2}}(\bar{i}y_1 - \bar{i}y_2)) \\ &= \bar{x_1}(\frac{1}{\sqrt{2}}(y_1 + y_2)) + \bar{x_2}(\frac{1}{\sqrt{2}}(-iy_1 + iy_2))\end{aligned}$$

Hence $T^*y = \frac{1}{\sqrt{2}}(y_1 + y_2, iy_1 - iy_1)$. By direct calculation, it is checked that $TT^*y = T^*Ty = y$ for all $y \in \mathbb{C}^2$. Hence T is unitary.

Q2. Let $x \in T^*(M_2^\perp)$, there exists $m_2 \in M_2^\perp$ with $T^*m_2 = x$. Let $m_1 \in M_1$, then

$$\begin{aligned}\langle m_1, x\rangle &= \langle m_1, T^*m_2\rangle \\ &= \langle Tm_1, m_2\rangle \\ &= 0\end{aligned}$$

as $Tm_1 \in M_2$ and $m_2 \in M_2^\perp$.