MMAT5010 2021 Assignment 7

Q1. Let $T : \ell_p \to \ell_p$ be defined by $Tx = (0, x(1), x(2), \dots)$. Recall the definition of an adjoint operator: $T^* : (\ell_p)^* \to (\ell_p)^*$ is defined by $x^* \mapsto T^*x^* \in (\ell_p)^*$, where $T^*x^*(x) = x^*(Tx)$. We will first see what is T^* in this case.

Fix $x^* \in (\ell_p)^*$. Then by the above:

$$(T^*x^*)(x) = x^*((0, x(1), x(2), \dots))$$

for any $x = (x(1), x(2), \dots) \in \ell_p$.

Now recall that $(\ell_p)^*$ is isometric isomorphic to ℓ_q. If $y \in \ell_q$, then y acts on elements x in ℓ_p via

$$y(x) = \sum_{i=1}^{\infty} y(i)x(i)$$

We now see T^* as a map from ℓ_q to ℓ_q. Using the above identification,

$$T^*y(x) = \sum_{i=1}^{\infty} y(i+1)x(i)$$

for every $y \in \ell_q$. We need to figure out what is $T^*y \in \ell_q$. To do this, observe that the actions of $(y(2), y(3), \dots)$ and T^*y are the same when acting on ℓ_p. Hence

$$T^*y = (y(2), y(3), \dots)$$

Q2. Let $x, y \in X$ such that $||x - y|| > c > 0$. By Hahn Banach Theorem there exists $f \in X^*$ such that $f(x - y) = ||x - y|| > c$. Hence $f(x) > c + f(y)$.