MMAT5010 2021 Assignment 1

Q1. ($\Rightarrow$) Suppose X is a Banach space, we want to show that S_X is complete. Let (x_n) be a Cauchy sequence in S_X. Because X is complete, (x_n) converges to some $x \in X$, because S_X is closed, so $x \in S_X$.

($\Leftarrow$) Suppose that S_X is complete. Let (x_n) be a Cauchy sequence in X, because

$$|\,||x_n|| - ||x_m||\,| \leq ||x_n - x_m||$$

it follows that $(||x_n||)$ is a Cauchy sequence in $\mathbb{R}$. Because $\mathbb{R}$ is complete, $(||x_n||)$ converges to some $L \in \mathbb{R}$. If $L = 0$, then $\lim_{n\to\infty} ||x_n - 0|| = 0$, i.e. $x_n \to 0$. If $L \neq 0$, we may assume that $x_n \neq 0$. then $(x_n/||x_n||)$ is a Cauchy sequence in S_X (**Fact:** if (λ_n) is a Cauchy sequence in $\mathbb{R}$, (x_n) a Cauchy sequence in X, then $(\lambda_n x_n)$ is a Cauchy sequence in X) By assumption, $x_n/||x_n||$ is convergent to some $s \in S_X$, so $x_n = ||x_n|| \frac{x_n}{||x_n||}$ is convergent to $Ls \in X$.

Q2. (a) We need to check

(i) If $q(x, y) = 0$, then $x = 0_X$ and $y = 0_Y$

(ii) $q(tx, ty) = tq(x, y)$ for $t \in \mathbb{R}$, $x \in X$, $y \in Y$

(iii) $q(x_1 + x_2, y_1 + y_2) \leq q(x_1, y_1) + q(x_2, y_2)$ for $x_1, x_2 \in X$, $y_1, y_2 \in Y$

For (i), suppose $q(x, y) = 0$, then $||x||_X + ||y||_Y = 0$, therefore $||x||_X = ||y||_Y = 0$, hence $x = 0_X$ and $y = 0_Y$.

For (ii), suppose $t \in \mathbb{R}$, $x \in X$, $y \in Y$, then $q(tx, ty) = ||tx||_X + ||ty||_Y = t(||x||_X + ||y||_Y) = tq(x, y)$

For (iii), let $x_1, x_2 \in X$, $y_1, y_2 \in Y$, then

$$q(x_1+x_2, y_1+y_2) = ||x_1+x_2||_X+||y_1+y_2||_Y \leq ||x_1||_X+||x_2||_X+||y_1||_Y+||y_2||_Y = q(x_1, y_1)+q(x_2, y_2)$$

(b) ($\Rightarrow$)Suppose that $X \oplus Y$ is a Banach space. By symmetry it suffices to show X is a Banach space. Let (x_n) be a Cacuhy sequence in X, then $||x_n||_X = q(x_n, 0_Y)$, so that $(x_n, 0_Y)$ is a Cauchy sequence in $X \oplus Y$.

By assumption, $\lim(x_n, 0_y) = (x, y)$ exists. Since $||x_n - x|| = q(x_n - x, 0_Y) \leq q(x_n - x, 0_Y - y)$, it follows that (x_n) converges to x.

($\Leftarrow$) Suppose both X, Y are Banach spaces. Let (x_n, y_n) be a Cauchy sequence in $X \oplus Y$, note $||x_n||_X \leq q(x_n, y_n)$, therefore, (x_n) is a Cauchy sequence in X, by assumption (x_n) converges to some $x \in X$, i.e. $\lim_n ||x - x_n||_X = 0$. Similarly, (y_n) is convergent to some $y \in Y$, i.e. $\lim_n ||y - y_n||_Y = 0$

So $\lim_n q(x_n - x, y_n - y) = 0$ and hence (x_n, y_n) converges to (x, y).