THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050A Tutorial 8

1. Show that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{Q}$ but discontinuous on $\mathbb{R}\backslash\mathbb{Q}$. (**Hints:** Write $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$. Use the continuity of f on $\mathbb{Q}$ and the density of $\mathbb{Q}$ to construct a nested sequence of closed bounded intervals I_n such that $r_n \notin I_{n+1}$ and that f is continuous on $\cap_{n=1}^{\infty} I_n$.)

2. If $f : [0, 1] \to \mathbb{R}$ is continuous and has only rational (respectively,irrational) values, must f be a constant?

3. Let $f : [0, 1] \to [0, 1]$ be continuous. Show that f has a fixed point. ($c \in [0, 1]$ is said to be fixed point of f is $f(c) = c$.)

4. Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be a (not necessarily continuous) function with the property that for every $x \in I$, the function f is bounded on a neighborhood $V_\delta(x)$ of x. Prove that f is bounded on I. Can the closedness condition be dropped?

5. Determine if the following functions are uniformly continuous:

   (a) $f(x) : (0, 1) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$,

   (b) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$,

   (c) $f : [0, M) \to \mathbb{R}$ defined by $f(x) = x^2$, where $M > 0$,

   (d) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$,

   (e) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x^2 + 1}$,

   (f) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(x^2)$.

Q1

Write $\mathbb{Q} \cap (0,1) = \{r_n\}_{n=1}^{\infty}$

f is continuous at $r_1 \Rightarrow \exists\ \delta_1 > 0$ s.t. $|f(x)-f(y)| < 1 \quad \forall\ x,y \in (r_1-\delta_1, r_1+\delta_1)$

We may take δ_1 small enough s.t. $(r_1-\delta_1, r_1+\delta_1) \subset (0,1)$

Let $n_2 := \min\{ n \in \mathbb{N} : r_n \in (r_1-\delta_1, r_1+\delta_1) \setminus \{r_1\} \}$.

We know that $n_2 > 1$.

f is continuous at $r_{n_2} \Rightarrow \exists\ \delta_2 > 0$ s.t. $|f(x)-f(y)| < \frac{1}{2} \quad \forall\ x,y \in (r_{n_2}-\delta_2, r_{n_2}+\delta_2)$

We may take δ_2 small enough s.t. $[r_{n_2}-\delta_2, r_{n_2}+\delta_2] \subset (r_1-\delta_1, r_1+\delta_1) \setminus \{r_1\}$

let $n_3 := \min\{ n \in \mathbb{N} : r_n \in (r_{n_2}-\delta_2, r_{n_2}+\delta_2) \setminus \{r_{n_2}\} \}$.

We know that $n_3 > n_2$ and we repeat the process.

In fact, we claim the following:

$\exists$ a sequence of open intervals$^{\neq\phi}$ $\{I_n\}_{n=1}^{\infty}$ (i.e. I_n are open intervals$^{\neq\phi}$) st.

① $I_1 \supset I_2 \supset I_3 \supset \cdots$, ② $r_k \notin \overline{I_k}$ (if the interval is (a,b), $\overline{(a,b)} := [a,b]$ $(b>a)$)

③ $|f(x)-f(y)| < \frac{1}{k} \quad \forall\ x,y \in I_k$

We assert that the sequence exists if "for each $\{I_1, \dots, I_N\}$ st. ① and ② and ③ hold, we can find I_{N+1} st. $\{I_1, I_2, \dots, I_{N+1}\}$ satisfies ① and ② and ③"

Given $\{I_1, I_2, \dots, I_N\}$ satisfying ①, ② and ③,

Let $m := \min\{ k \in \mathbb{N} : r_k \in I_N \setminus \{r_{N+1}\} \}$, then $m > N+1$ (why?)

$\exists\ \delta_{N+1} > 0$ st. $|f(x)-f(y)| < \frac{1}{N+1} \quad \forall\ x,y \in (r_m-\delta_{N+1}, r_m+\delta_{N+1})$

We can take δ_{N+1} small enough st. $[r_m-\delta_{N+1}, r_m+\delta_{N+1}] \subset I_N \setminus \{r_{N+1}\}$.

Then take $I_{N+1} := (r_m-\delta_{N+1}, r_m+\delta_{N+1})$

You can omit assertion by selecting δ_{N+1} according to some specific rules.

By the assertion, we obtain the sequence. Now $\bigcap_{k=1}^{\infty} \overline{I_k} \neq \phi$ by Nested interval Thm.

But $r_k \notin \overline{I_k} \Rightarrow$ Every element in $\bigcap_{k=1}^{\infty} \overline{I_k}$ is irrational number.

$\bigcap_{k=1}^{\infty} \overline{I_k}$ should be singleton because $\bigcap_{k=1}^{\infty} \overline{I_k} = [a,b]$ for some a,b but $\mathbb{Q} \cap [a,b] = \phi$

$\therefore\ a = b \in \mathbb{R} \setminus \mathbb{Q}$ and $\bigcap_{k=1}^{\infty} \overline{I_k}$ is singlton.

Doesn't matter if $\bigcap_{k=1}^{\infty} \overline{I_k}$ is singlton or not, take $r \in \bigcap_{k=1}^{\infty} \overline{I_k}$

(by ③) f is continuous at r if $r \in \bigcap_{k=1}^{\infty} I_k$. Let's assume for each open interval $I_k = (a_k, b_k)$, its boundary pts a_k, b_k are rationals, which can be done in construction of seq $\{I_k\}$.