THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2019-20
Homework 6
Due Date: 7th November 2019

Compulsory Part

1. Find the units in the following rings:
   (a) $\mathbb{Z}$.
   (b) The ring R of all real valued functions on $\mathbb{R}$.
   (c) $R[x]$ where R is an integral domain.

2. Show that the set $R^{\times}$ of units in a ring R forms a group under multiplication.

3. Let R be an integral domain. Show that the polynomial ring $R[x]$ is also an integral domain.

Optional Part

1. Show that $a^2 - b^2 = (a+b)(a-b)$ for all a, b in a ring R if and only if R is commutative.

2. A ring R such that $a^2 = a$ for any $a \in R$ is called a **Boolean ring**. Show that every Boolean ring is commutative.

3. Let R be a commutative ring. Show that the **binomial theorem** holds, i.e.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

   for any $a, b \in R$ and for any positive integer n.

4. Let R be the set of all real-valued functions f on $\mathbb{R}$ such that $f(0) = 0$. Let $+$ and $\cdot$ be the usual addition and multiplication operations for functions.
   (a) Show that $f + g \in R$ for all $f, g \in R$.
   (b) Show that $f \cdot g \in R$ for all $f, g \in R$.
   (c) With respect to $+$, what is the additive identity element of R, if it exists?
   (d) With respect to $\cdot$, what is the multiplicative identity element of R, if it exists?

5. (a) Is the product of two units in a ring necessarily a unit? If so, prove it; if not, provide a counterexample.
   (b) Is the sum of two units in a ring necessarily a unit? If so, prove it; if not, provide a counterexample.

6. Verify that under the convention $\deg 0 = -\infty$, the following rules hold for all polynomials $f, g \in R[x]$ where R is an integral domain:

(a) $\deg(fg) = \deg f + \deg g$.

(b) $\deg(f \pm g) \leq \max\{\deg f, \deg g\}$.

7. **Definition.** Let R be a ring. A subset S of R is said to be a **subring** of R if it is a ring under the addition $+$ and multiplication $\cdot$ associated with R, and its additive and multiplicative identity elements 0, 1 are those of R.

   To show that a subset S of a ring R is a subring, it suffices to show that:

   - $1_R \in S$,
   - $a - b \in S$ for any $a, b \in S$, and
   - S is closed under multiplication: $a \cdot b \in S$ for all $a, b \in S$.

   The **center** $Z(R)$ of R is defined as follows:

$$Z(R) = \{r \in R : rs = sr \text{ for all } s \in R\}.$$

   Show that $Z(R)$ is a subring of R.

8. Let D be an integral domain. If there exists a positive integer n such that $na = \overbrace{a + \cdots + a}^{n \text{ times}} = 0$ for any $a \in D$, then D is said to be of **finite characteristic**; in this case, we define the **characteristic** of D to be

$$\text{char}(D) := \min\{n \in \mathbb{Z}_{>0} \mid na = 0\ \forall a \in D\}.$$

   If no such positive integer exists, we say that D is of **characteristic 0**, denoted as $\text{char}(D) = 0$.

   (a) Show that if $n1 \neq 0$ for any $n \in \mathbb{Z}_{>0}$, then D is of characteristic 0; otherwise, we have

$$\text{char}(D) = \min\{n \in \mathbb{Z}_{>0} \mid n1 = 0\ \forall a \in D\}.$$

   (b) Hence show that the characteristic of an integral domain is either 0 or a prime.