TUTORIAL 1

Ex. 1: l^p-spaces ($1 \le q \le \infty$). Consider linear vector space:

$$X := \mathbb{R}^n(\text{ or } \mathbb{C}^n) := \{(x_1, x_2, \cdots, x_n), \quad x_i \in \mathbb{R}(\text{ or } \mathbb{C})\}.$$

Define

$$|x|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

Show that $(X, |\cdot|_p)$ is a complete normed space (Banach space).

Proof. To show $(X, |\cdot|_p)$, we need: 1. to show $(X, |\cdot|_p)$ is a normed space (by definition, three laws.) 2. to show its completeness, that is, to show each Cauchy sequence has a limit in $(X, |\cdot|_p)$.

Step 1. We justify that l^p is a normed space in this step. The only non-trivial thing is to justify the triangular inequality for $1 \le p < \infty$, which is a direct consequence of the following Minkowski's inequality:

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |y_i|^p\right)^{1/p}, \tag{0.1}$$

for any $x, y \in \mathbb{R}^n$ or $\mathbb{C}^n$.

Proof of (0.1):

$$\begin{aligned}
\sum_{i=1}^n |x_i + y_i|^p &\le \sum_{i=1}^n |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i||x_i + y_i|^{p-1} \\
&\le \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \cdot \left(\sum_{i=1}^n |x_i + y_i|^{(p-1)p_*}\right)^{1/p_*} \\
&\quad + \left(\sum_{i=1}^n |y_i|^p\right)^{1/p} \cdot \left(\sum_{i=1}^n |x_i + y_i|^{(p-1)p_*}\right)^{1/p_*} \quad (\frac{1}{p} + \frac{1}{p_*} = 1) \\
&\le (|x|_p + |y|_p)\, |x + y|_p^{p-1}, \qquad (0.2)
\end{aligned}$$

where we have used the following Holder's inequality

$$\sum_{i=1}^n |x_i y_i| \le \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \cdot \left(\sum_{i=1}^n |y_i|^{p_*}\right)^{1/p_*}$$

in the second line. Then for $x + y \ne 0$, (0.1) follows from dividing both side of (0.2) by $|x + y|_p^{p-1}$. If $x + y = 0$, (0.1) is trivial.

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Step 2. Completeness. Take any Cauchy sequence $\{x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \cdots x_n^{(k)})\}_{k=1}^{\infty}$. By definition, for each $\varepsilon > 0$, there exists a positive constant $N > 0$, such that for any $k, m > N$, one has

$$\left(\sum_{j=1}^{n} |x_j^{(k)} - x_j^{(m)}|^p\right)^{1/p} < \varepsilon, \tag{0.3}$$

which implies that for each j, $\{x_j^{(k)}\}_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$ or $\mathbb{C}$. By the completeness of $\mathbb{R}$ or $\mathbb{C}$, there exists x_j, such that

$$\lim_{k\to\infty} x_j^{(k)} = x_j.$$

Then by taking $k \to \infty$ in (0.3), one has, for any $m > N$, that

$$\left(\sum_{j=1}^{n} |x_j - x_j^{(m)}|^p\right)^{1/p} < \varepsilon.$$

This proves that $\{x^{(k)}\}$ converges to $\{x = (x_1, \cdots, x_n)\}$. Therefore, $(\mathbb{R}^n, |\cdot|_p)$ is a complete normed space. $\square$

Ex. 2: $X = C(K)$ (Real-value continuous functions), where K is a compact subset in $\mathbb{R}^n$. Define

$$\|f\|_{C(K)} = \sup_{x\in K} |f(x)|.$$

Show that $(X, \|\cdot\|_{C(K)})$ is a Banach space.

Proof. **Step 1.** Show that $(X, \|\cdot\|_{C(K)})$ is a normed space. (Trivial)
Step 2. Justification of completeness. Let f_n be a Cauchy sequence in X. By definition, for each $\varepsilon > 0$, there exists a positive constant $N > 0$, such that for any $n, k > N$ and $x \in K$, it holds that

$$|f_n(x) - f_k(x)| < \varepsilon. \tag{0.4}$$

Hence for each $x \in K$, $f_n(x)$ is Cauchy in $\mathbb{R}$. Then similar as in Ex. 1, there exists a function $f(x)$, such that $f_n(x)$ converges to $f(x)$ pointwisely in K. Since N is independent of x, we can take $k \to \infty$ in (0.4). Hence, f_n uniformly converges to f. Therefore, $f \in C(K)$ by compactness of K. This proves that $(X, \|\cdot\|_{C(K)})$ is a Banach space. $\square$

Ex. 3. Let $K = [0, 1]$. Define a norm $\|f\|_1 := \int_0^1 |f(x)|\mathrm{d}x$. Then $(C[0, 1], \|\cdot\|_1)$ is not Banach.

Proof. To show $(C[0, 1], \|\cdot\|_1)$ is not Banach, we need to construct a counterexample. More precisely0, we need to construct a Cauchy sequence that has no limit in $(C[0, 1], \|\cdot\|_1)$. Consider the following sequence:

$$f_n(x)|_{n\geq 2} = \begin{cases} 0, \quad 0 \leq x \leq 1/2, \\ n(x - 1/2), \quad 1/2 < x \leq 1/2 + 1/n, \\ 1, \quad 1/2 + 1/n < x \leq 1. \end{cases}$$

Then

$$\|f_n - f_m\|_1 \leq \frac{1}{2n} + \frac{1}{2m} \to 0,$$

as $m, n \to \infty$. Hence $\{f_n\}_{n\geq 2}$ is a Cauchy sequence in $(C[0,1], \|\cdot\|_1)$. Let

$$f(x) = \begin{cases} 0, & 0 \leq x \leq 1/2, \\ 1, & 1/2 < x \leq 1. \end{cases}$$

Then we have

$$\|f_n - f\|_1 = \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} (1 - n(x - \frac{1}{2}))\mathrm{d}x = \frac{1}{2n} \to 0,$$

as $n \to \infty$. Clearly $f(x) \notin C[0,1]$. Therefore, $(C[0,1], \|\cdot\|_1)$ is not Banach. $\square$