Review on Week 5

Cauchy Sequences

The definition of a Cauchy sequence is similar to the definition of a convergent sequence.

Definition (c.f. Definition 3.5.1)**.** A sequence of real numbers (x_n) is said to be a *Cauchy sequence* if for every $\varepsilon > 0$, there exist a natural number N such that

$$|x_n - x_m| < \varepsilon, \qquad \forall n, m \geq N.$$

Example 1 (c.f. Example 3.5.2)**.** $(1/n)$ is a Cauchy sequence.

Solution. We need to show that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon, \qquad \forall n, m \geq N.$$

Note that if $n, m \geq N$,

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|-\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}.$$

Let $\varepsilon > 0$. By **Archimedean Property**, there exists $N \in \mathbb{N}$ such that $1/N < \varepsilon/2$. Then for any $n, m \geq N$,

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{2}{N} < \varepsilon.$$

Example 2. $(1 - (-1)^n)$ is not a Cauchy sequence.

Solution. We need to show that there exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$, there exists $n, m \geq N$ such that

$$|(1 - (-1)^n) - (1 - (-1)^m)| \geq \varepsilon.$$

Note that the sequence is alternating between 0 and 2, hence any successive terms have difference 2. Take $\varepsilon = 2 > 0$. Then for any $N \in \mathbb{N}$, take $n = N + 1$ and $m = N$. Then

$$|(1 - (-1)^n) - (1 - (-1)^m)| = 2 \geq \varepsilon.$$

The following lemmas tell us some relations between convergent and Cauchy sequences.

Lemma (c.f. Lemma 3.5.3)**.** *A convergent sequence of real numbers is a Cauchy sequence.*

Lemma (c.f. Lemma 3.5.4)**.** *A Cauchy sequence of real numbers is bounded.*

The following theorem tell us that Cauchy sequences and convergent sequences are actually equivalent! This is also a main theorem of this course and is due to **Bolzano-Weierstrass Theorem**. They all come from the **Completeness Property of** $\mathbb{R}$.

Cauchy Convergence Criterion (c.f. 3.5.5)**.** *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Remark. By Lemma 3.5.3, a sequence being Cauchy is a formally weaker condition than being convergent. It is easier for us to check whether or not a sequence is Cauchy than checking its convergence because we don't need to specify the limit.

Applications

Let's present two examples to show that the **Cauchy Convergence Criterion** is useful.

Example 3 (c.f. Example 3.5.6(b))**.** The following sequence is convergent:

$$\left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n+1}}{n!}\right)$$

Solution. We cannot prove this by definition in a elementary way because the limit is $1-1/e$. Using **Cauchy Convergence Criterion**, it suffices to show that this sequence is Cauchy.

Let y_n be the n-th term of this sequence. Note that if $n \geq m \geq N$,

$$\begin{aligned} |y_n - y_m| &= \left|\frac{(-1)^{m+2}}{(m+1)!} + \frac{(-1)^{m+3}}{(m+2)!} + \cdots + \frac{(-1)^{n+1}}{n!}\right| \\ &\leq \frac{1}{(m+1)!} + \frac{1}{(m+2)!} + \cdots + \frac{1}{n!} \\ &\leq \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}} \\ &< \frac{1}{2^m}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) \\ &\leq \frac{1}{2^N} \cdot 2 = \frac{1}{2^{N-1}} \end{aligned}$$

Let $\varepsilon > 0$. By **Archimedean Property** (How?), choose N such that $1/2^{N-1} < \varepsilon$. Then

$$|y_n - y_m| < \frac{1}{2^{N-1}} < \varepsilon, \qquad \forall n, m \geq N.$$

Therefore (y_n) is a Cauchy sequence.

Example 4 (c.f. Example 3.5.6(c))**.** The harmonic series $(1 + 1/2 + 1/3 + \cdots)$ diverges.

Solution. Again, it is tedious for us to check by definition that the series does not converge to any real numbers. Hence we simply check that this sequence is not Cauchy.

Let h_n be the n-th term of this sequence. For $n \geq m$, note that

$$|h_n - h_m| = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n} \geq \frac{n-m}{n} = 1 - \frac{m}{n}.$$

Hence if we take $\varepsilon = 1/2 > 0$, for any $N \in \mathbb{N}$, take $n = 2N$ and $m = N$. Then

$$|h_n - h_m| \geq 1 - \frac{m}{n} = 1/2 = \varepsilon.$$

Quiz 1 on Next Thursday 17/10

Please be reminded that there will be a quiz next week. It will start at 9:30 a.m., during lecture. Please be punctual.

Exercises

Question 1 (c.f. Section 3.5, Ex.5)**.** If $x_n = \sqrt{n}$, show that (x_n) satisfies $\lim |x_{n+1} - x_n| = 0$, but it is not a Cauchy sequence.

Solution. Direct calculation gives

$$\begin{aligned}\lim_{n\to\infty} |x_{n+1} - x_n| &= \lim_{n\to\infty} |\sqrt{n+1} - \sqrt{n}| \\ &= \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \lim_{n\to\infty} \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ &= 0\end{aligned}$$

To show that this sequence is not Cauchy, take $\varepsilon = 1 > 0$. For any $N \in \mathbb{N}$, take $n = (N+1)^2$ and $m = N^2$, then

$$|x_n - x_m| = |\sqrt{(N+1)^2} - \sqrt{N^2}| = |N + 1 - N| = 1 \geq \varepsilon.$$

Remark. This exercise tells us that we cannot look at the limit of the difference between successive terms only. We need to consider the difference between **every** terms after some large numbers N. However, the next exercise tell us that we can do so if we impose a stronger condition on the difference between successive terms.

Question 2 (c.f. Section 3.4, Ex.9)**.** If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.

Solution. Note that by triangle inequality, for $n \geq m$,

$$|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|.$$

Hence for any $n \geq m \geq N$,

$$\begin{aligned}|x_n - x_m| &\leq r^{n-1} + r^{n-2} + \cdots + r^m \\ &< r^m \cdot (1 + r + r^2 + \cdots) \\ &< r^N \cdot \frac{1}{1-r}\end{aligned}$$

Let $\varepsilon > 0$. We want to find $N \in \mathbb{N}$ by Archimedean Property such that

$$r^N \cdot \frac{1}{1-r} < \varepsilon.$$

Solving the inequality for N yields $N > \dfrac{\ln \varepsilon + \ln(1-r)}{\ln r}$. We are done after we choose such N by Archimedean Property.

Question 3 (c.f. Section 3.4, Ex.10)**.** If $x_1 < x_2$ are arbitrary real numbers and

$$x_n = \frac{1}{2}(x_{n-2} + x_{n-1}) \qquad \text{for } n > 2,$$

show that (x_n) is convergent. What is its limit?

Solution. To show that the sequence is convergent, we show that it is Cauchy. Note that since the next term is constructed by averaging,

$$|x_{n+1} - x_n| = \frac{x_2 - x_1}{2^{n-1}}, \qquad \forall n \in \mathbb{N}.$$

(Prove this by induction!) Hence for $n \geq m \geq N$,

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\ &= \frac{x_2 - x_1}{2^{n-2}} + \frac{x_2 - x_1}{2^{n-3}} + \cdots + \frac{x_2 - x_1}{2^{m-1}} \\ &< \frac{x_2 - x_1}{2^{m-1}} \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) \\ &\leq \frac{x_2 - x_1}{2^{N-1}} \cdot 2 \\ &= \frac{x_2 - x_1}{2^{N-2}} \end{aligned}$$

Let $\varepsilon > 0$. Then by Archimedean Property, choose $N \in \mathbb{N}$ such that

$$\frac{x_2 - x_1}{2^{N-2}} < \varepsilon.$$

Then we can show that (x_n) is Cauchy and hence convergent.

To find the limit of this sequence, note that we cannot find the limit by solving

$$x = \frac{1}{2}(x + x).$$

Instead, since the sequence is convergent, all of its subsequence will also converger to the same limit. In particular, we fing the general term of the subsequence x_{2n+1}. Observe that

$$x_{2n+1} = x_1 + \frac{x_2 - x_1}{2}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^{n-1}}\right) = x_1 + \frac{x_2 - x_1}{2}\sum_{k=0}^{n-1}\frac{1}{4^k}, \quad \forall n \in \mathbb{N}.$$

(Prove this by induction!) Hence we can calculate the limit of (x_n) by

$$\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(x_{2n+1}) = x_1 + \frac{x_2 - x_1}{2}\sum_{k=0}^{\infty}\frac{1}{4^k} = \frac{1}{3}x_1 + \frac{2}{3}x_2.$$