# Review on Week 4

## Nested Interval

The following theorems are not hard to be observed and visualized. They are based on the **Completeness Property of** $\mathbb{R}$, and are essential to prove the **Bolzano-Weierstrass Theorem**, which is one of the main theorems in this course.

**Nested Intervals Property** (c.f. 2.5.2). *If $I_n = [a_n, b_n]$ is a nested sequence of closed bounded intervals for each $n \in \mathbb{N}$, i.e.,*

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

*then there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$.*

**Remark.** The condition $I_n$ is **closed** for all $n \in \mathbb{N}$ cannot be dropped. (See Exercise!)

If we impose one more condition on the lengths of the intervals, we have

**Theorem** (c.f. Theorem 2.5.3). *If $I_n = [a_n, b_n]$ is a nested sequence of closed bounded intervals for each $n \in \mathbb{N}$ such that the lengths $b_n - a_n$ of $I_n$ satisfy*

$$\inf\{b_n - a_n : n \in \mathbb{N}\} = 0,$$

*then the number $\xi \in \mathbb{R}$ contained in $I_n$ for all $n \in \mathbb{N}$ is unique.*

## Subsequences

**Definition** (c.f. Definition 3.4.1). Let $(x_n)$ be a sequence of real numbers and let

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

be a strictly increasing sequence of natural numbers. Then

$$(x_{n_k}) = (x_{n_1}, x_{n_2}, ..., x_{n_k}, x_{n_{k+1}}, ...)$$

is called a *subsequence* of $(x_n)$.

**Remark.** The original sequence is indexed by $n$ while the subsequence is indexed by $k$, the $n$ plays no role in the subsequence.

**Example 1.** Let $(x_n)$ be a sequence.

- If we take $n_k = 2k$ for each $k \in \mathbb{N}$, we have the subsequence $(x_{n_k}) = (x_2, x_4, x_6, ...)$.
- If we take $n_k = k+10$ for each $k \in \mathbb{N}$, we have the subsequence $(x_{n_k}) = (x_{11}, x_{12}, x_{13}, ...)$.
- If we take $(n_k) = (1, 10, 100, ...)$ for each $k \in \mathbb{N}$, we have the subsequence

$$(x_{n_k}) = (x_1, x_{10}, x_{100}, ...).$$

**Theorem** (c.f. Theorem 3.4.2). *Let $(x_n)$ be a sequence of real numbers that converges to $x \in \mathbb{R}$. Then every subsequence $(x_{n_k})$ of $(x_n)$ also converges to $x$.*

**Example 2** (c.f. Section 3.4, Ex.7)**.** Establish the convergence and find the limits of the following sequences:

(a) $\left((1+1/n^2)^{n^2}\right)$

(b) $((1+1/2n)^n)$

(c) $\left((1+1/n^2)^{2n^2}\right)$

(d) $((1+2/n)^n)$

**Solution.** We need to use the following fact about the exponential function:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n, \quad \text{where } x \in \mathbb{R}$$

(a) Note that if we take $m = n^2$,

$$\left(1+\frac{1}{n^2}\right)^{n^2} = \left(1+\frac{1}{m}\right)^m.$$

Hence it is a subsequence of the sequence $((1+1/n)^n)$, which converges to $e^1 = e$. Hence this sequence also converges to $e$.

(b) Note that

$$\left(1+\frac{1}{2n}\right)^n = \left(1+\frac{1/2}{n}\right)^n.$$

Hence this sequence converges to $e^{1/2} = \sqrt{e}$.

(c) Note that if we take $m = 2n^2$,

$$\left(1+\frac{1}{n^2}\right)^{2n^2} = \left(1+\frac{2}{m}\right)^m.$$

Hence it is a subsequence of the sequence $((1+2/n)^n)$, which converges to $e^2$. Hence this sequence also converges to $e^2$.

(d) Trivial, it converges to $e^2$.

**Theorem** (c.f. Theorem 3.4.4)**.** *Let $(x_n)$ be a sequence of real numbers. Then the following statements are equivalent:*

*(i) The sequence $(x_n)$ does not converges to $x \in \mathbb{R}$.*

*(ii) There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists natural number $n_k > k$ such that $|x_{n_k} - x| \geq \varepsilon_0$.*

*(iii) There exists an $\varepsilon_0 > 0$ and a subsequence $(x_{n_k})$ of $(x_n)$ such that $|x_{n_k} - x| \geq \varepsilon_0$.*

*(iv) There exists a subsequence $(x_{n_k})$ of $(x_n)$ that does not converge to $x$.*

**Divergence Criteria.** *If a sequence $(x_n)$ of real numbers has either of the following properties, then $(x_n)$ is divergent.*

*(i)* $(x_n)$ *has two convergent subsequences* $(x_{n_k})$ *and* $(x_{r_k})$ *whose limits are not equal.*

*(ii)* $(x_n)$ *is unbounded.*

**Example 3** (c.f. Examples 3.4.6)**.** The sequence $(x_n) = ((-1)^n)$ is divergent. If we consider the subsequences

$$(x_{n_k}) = (x_{2k}) = (1, 1, 1, ...) \quad \text{and} \quad (x_{r_k}) = (x_{2k-1}) = (-1, -1, -1, ...),$$

their limits are 1 and $-1$ respectively, hence $(x_n)$ is divergent.

## The Bolzano-Weierstrass Theorem

Having the notion of subsequences, we can state another main theorem of this course, which relies on the **Nested Interval Property**.

**The Bolzano-Weierstrass Theorem.** *A bounded sequence of real numbers has a convergent subsequence.*

# Exercises

**Question 1** (c.f. Section 2.5, Ex.8)**.** Let $J_n = (0, 1/n)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} J_n = \emptyset$.

**Solution.** To show that a set is empty, suppose that it contains some elements and find contradictions.

Suppose that $x \in \bigcap_{n=1}^{\infty} J_n$. Since $x \in J_1 = (0, 1)$, so $x > 0$. By Archimedian Property, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{N} < x.$$

But this contradict to $x \in J_N = (0, 1/N)$.

**Question 2** (c.f. Section 3.4, Ex.4)**.** Determine the convergence of the following sequences.

(a) $(1 - (-1)^n + 1/n)$, (b) $(\sin(n\pi/4))$, (c) $(\sin(n\pi)/4)$.

**Solution.** We will apply divergence criterion on the first two sequences.

(a) Obviously, we should take the odd terms and even terms as the subsequences. Consider the subsequences $(x_{2n})$ and $(x_{2n-1})$. Note that

$$\lim(x_{2n}) = \lim\left(1 - (-1)^{2n} + \frac{1}{2n}\right) = \lim\left(\frac{1}{2n}\right) = 0$$
$$\lim(x_{2n-1}) = \lim\left(1 - (-1)^{2n-1} + \frac{1}{2n-1}\right) = \lim\left(2 + \frac{1}{2n-1}\right) = 2$$

(b) Note that the sine function is $2\pi$-periodic. So when $n$ is increased by 8, the terms are the same. Consider the subsequences $(x_{8n})$ and $(x_{8n+1})$. Note that

$$\lim(x_{8n}) = \lim\left(\sin\left(\frac{8n\pi}{4}\right)\right) = \lim(\sin(2n\pi)) = 0$$
$$\lim(x_{8n+1}) = \lim\left(\sin\left(\frac{(8n+1)\pi}{4}\right)\right) = \lim\left(\sin\left(2n\pi + \frac{\pi}{4}\right)\right) = \sin\frac{\pi}{4}$$

(c) Note that $\sin(n\pi) = 0$ for any $n \in \mathbb{N}$. Hence this sequence is a constant zero sequence. Thus it is convergent to zero.

**Question 3** (c.f. Section 3.4, Ex.9)**.** Suppose that every subsequence of $(x_n)$ has a subsequence that converges to 0. Show that $\lim(x_n) = 0$.

**Solution.** Suppose on a contrary that $\lim(x_n) \neq 0$. Hence there exists $\varepsilon_0 > 0$ and a subsequence $(x_{n_k})$ of $(x_n)$ such that

$$|x_{n_k} - 0| = |x_{n_k}| \geq \varepsilon_0, \quad \forall k \in \mathbb{N}. \tag{1}$$

By the assumption, there is a subsequence $(x_{n_{k_j}})$ of $(x_{n_k})$ such that $\lim_j(x_{n_{k_j}}) = 0$. i.e., there exists $J \in \mathbb{N}$ such that

$$|x_{n_{k_j}} - 0| = |x_{n_{k_j}}| < \varepsilon_0 \quad \forall j \geq J.$$

In particular, take $K = k_J$, then it is a contradiction to (1) because $|x_{n_K}| < \varepsilon_0$.

**Question 4** (c.f. Section 3.4, Ex.14)**.** Let $(x_n)$ be a bounded sequence and let $s = \sup(x_n)$. Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of $(x_n)$ that converges to $s$.

**Solution.** We need to pick the subsequence $(x_{n_k})$ by choosing $n_k$ for each $k$. Firstly, by definition of supremum, take $n_1 \in \mathbb{N}$ such that

$$x_{n_1} > s - 1.$$

Then we claim that there exists an integer $n_2 > n_1$ such that

$$x_{n_2} > s - \frac{1}{2}.$$

Suppose not. Then $x_n \leq s - 1/2$ for any $n > n_1$. Hence the supremum of $(x_n)$ must be the maximum among $x_1, x_2, ..., x_{n_1}$ or $s - 1/2$, which must not be $s$ because $s \notin \{x_n : n \in \mathbb{N}\}$. In the similar way, we can pick $n_{k+1} > n_k$ inductively such that

$$x_{n_{k+1}} > s - \frac{1}{k+1}.$$

It follows by Squeeze Theorem that $\lim(x_{n_k}) = s$ because

$$s - \frac{1}{k} < x_{n_k} \leq s.$$