THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2019)
Suggested Solution of Homework 4: p.84: 4, 12, 19

4. Show that the following sequences are divergent.

   (a) $(1-(-1)^n+1/n)$, (2 marks)

   (b) $\sin(n\pi/4)$. (2 marks)

   Solution. For each question, we will consider two subsequences from the original sequence. These two subsequences converge to different limits, so by divergence criterion, the given sequence is divergent.

   (a) Let $x_n = (1-(-1)^n+1/n)$. Let $n_k = 2k$ and $m_k = 2k+1$ for $k \in \mathbb{N}$. The subsequence $x_{n_k} = (1/2k)$ converges to 0, while the subsequence $x_{m_k} = 2+1/(2k+1)$ converges to 2. Therefore, the sequence (x_n) itself is divergent.

   (b) Let $x_n = \sin(n\pi/4)$. Let $n_k = 2+8k$ and $m_k = 8k$ for $k \in \mathbb{N}$. The limit of subsequence $(x_{n_k} = \sin(\pi/2+2k\pi) = 1)$ is 1, while the other subsequence $(x_{m_k} = \sin(2k\pi) = 0)$ has limit equal to 0. These show that (x_n) is divergent.

12. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim_{k\to\infty}(1/x_{n_k}) = 0$. (3 marks)

    Solution. Since (x_n) is unbounded, for each $k \in \mathbb{N}$, there is some $m_k \in \mathbb{N}$ such that $|x_{m_k}| > k$. To construct a subsequence, we need to observe a seemingly but not really stronger conclusion. That is, for each $k \in \mathbb{N}$, there are infinitely many possible $m \in \mathbb{N}$ satisfying $|x_m| > k$. We now see why this claim is true.

    Suppose not, there are finitely many possible $m \in \mathbb{N}$ with the property that $|x_m| > k$. We can list them out, say $\{x_{m_1}, \dots, x_{m_l}\}$. Now, the sequence (x_n) is bounded. Indeed, $|x_n| \le \max\{k, |x_{m_1}|, \dots, |x_{m_l}|\}$ for each $n \in \mathbb{N}$. This contradicts the assumption, and hence the claim is true.

    In the following, we will construct a subsequence of x_n by the claim. That is to select $n_1, n_2, \dots$ with $n_{k+1} > n_k$. We will do it by induction. Let

    $$n_1 = 1 \quad \text{and} \quad n_k := \min\{n \in \mathbb{N} : n > n_{k-1}, |x_n| > k\} \text{ for } k \ge 2.$$

    The claim tells us that the set $\{n \in \mathbb{N} : n > n_{k-1}, |x_n| > k\}$ is nonempty and thus the induction works.

    Finally, since $|x_{n_k}| > k$ for $k \ge 2$, it is easy to see that $\lim_{k\to\infty}(1/x_{n_k}) = 0$.

19. Show that if (x_n) and (y_n) are bounded sequences, then

    $$\limsup(x_n + y_n) \le \limsup(x_n) + \limsup(y_n).$$

    Give an example in which the two sides are not equal. (3 marks)

Solution. Recall that for a sequence (x_n), $\limsup x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$, which is the limit of the sequence $(\sup_{k\geq n} x_k)_{n\in\mathbb{N}}$ indexed by n.

For each fixed $n \in \mathbb{N}$, we see that

$$\sup_{k\geq n}(x_k + y_k) \leq \sup_{k\geq n} x_k + \sup_{k\geq n} y_k, \tag{1}$$

because for each $i \geq n$, we have $x_i + y_i \leq \sup_{k\geq n} x_k + \sup_{k\geq n} y_k$ and RHS is an upper bound of the set $\{x_k + y_k : k \geq n\}$.

For any bounded sequences, its limit superior exists. In particular, $(x_n + y_n)$ is a bounded sequence. We can take limit on both sides of (1) and this gives the desired inequality.

For an example in which the two sides are not equal, we may put $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Both of them have limit superior equal to 1. Note that $x_n + y_n = 0$. LHS of the inequality is 0, while the RHS is 2.