THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2019)
Suggested Solution of Homework 2: p.61: 5(a), (c), 11; p.70: 9

5. Use the definition of the limit of a sequence to establish the following limits.

(3 marks each)

(a) $\lim_{n\to\infty}\left(\dfrac{n}{n^2+1}\right)=0$,

(c) $\lim_{n\to\infty}\left(\dfrac{3n+1}{2n+5}\right)=\dfrac{3}{2}$.

Solution:
(a) Notice that

$$\left|\frac{n}{n^2+1}-0\right|=\frac{n}{n^2+1}\leq\frac{n}{n^2}=\frac{1}{n}.$$

Let $\epsilon>0$. By Archimedean property, there is some $N\in\mathbb{N}$ such that $\dfrac{1}{N}<\epsilon$. For any $n\geq N$, we have

$$\left|\frac{n}{n^2+1}-0\right|\leq\frac{1}{n}\leq\frac{1}{N}<\epsilon.$$

By the definition of the limit of a sequence, we have $\lim_{n\to\infty}\left(\dfrac{n}{n^2+1}\right)=0$.

(c) Notice that

$$\left|\frac{3n+1}{2n+5}-\frac{3}{2}\right|=\left|\frac{(6n+2)-(6n+15)}{2(2n+5)}\right|=\frac{13}{2}\frac{1}{2n+5}\leq\frac{13}{2(2n)}=\frac{13}{4n}.$$

Let $\epsilon>0$. By Archimedean property, there is some $N\in\mathbb{N}$ such that $\dfrac{1}{N}<\dfrac{4\epsilon}{13}$. Now if $n\geq N$, we have

$$\left|\frac{3n+1}{2n+5}-\frac{3}{2}\right|\leq\frac{13}{4n}\leq\frac{13}{4N}<\epsilon.$$

This shows that $\lim_{n\to\infty}\left(\dfrac{3n+1}{2n+5}\right)=\dfrac{3}{2}$.

11. Show that $\lim_{n\to\infty}\left(\dfrac{1}{n}-\dfrac{1}{n+1}\right)=0$.

Solution:
Notice that

$$\left|\frac{1}{n}-\frac{1}{n+1}-0\right|=\frac{1}{n(n+1)}=\frac{1}{n^2+n}\leq\frac{1}{n}$$

Let $\epsilon > 0$. By Archimedean property, there is some $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \epsilon$. For any $n \geq N$, we have

$$\left| \frac{1}{n} - \frac{1}{n+1} - 0 \right| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

This shows that $\lim\limits_{n\to\infty} \left(\dfrac{1}{n} - \dfrac{1}{n+1} \right) = 0$.

9. Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n} y_n)$ converges. Find the limit.

(4 marks)

Solution:
Notice that

$$y_n = \sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \cdot \left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

hence

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n+1}} \leq \sqrt{n} y_n \leq \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n}} = \frac{1}{2}.$$

Let $b_n := \dfrac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n+1}} = \dfrac{1}{2}\sqrt{\dfrac{n}{n+1}} = \dfrac{1}{2}\sqrt{1 - \dfrac{1}{n+1}}$.

By **3.2.3 Theorem (a), 3.2.10 Theorem** and the result $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, we obtain $\lim\limits_{n\to\infty} b_n = \dfrac{1}{2}$. By squeeze theorem, we conclude that $(\sqrt{n} y_n)$ converges to $\dfrac{1}{2}$.