THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2019)
Suggested Solution of Homework 1: p.44: 2, 4, 5, 3

2. If $S := \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$. (3 marks)

   Solution:
   We propose that $\inf S = -1$ and $\sup S = 1$. First we show that $\inf S = -1$.

   -1 is a lower bound for the set S: for every $n \in \mathbb{N}$, $1/n \geq 0$. For every $m \in \mathbb{N}$, $1/m \leq 1$. Combining these two inequalities, we have $1/n - 1/m \geq -1$ for every $n, m \in \mathbb{N}$.

   Let $\epsilon > 0$. It suffices to show that $-1 + \epsilon$ fails to be a lower bound of S: By Archimedean property, there is some $n_0 \in \mathbb{N}$ such that $1/n_0 < \epsilon$. Taking $m = 1$, we have $1/n_0 - 1 < \epsilon - 1$. But $1/n_0 - 1$ is an element in S. It shows that $-1 + \epsilon$ is not a lower bound of S.

   Notice that $(-S) = \{-1/n + 1/m : n, m \in \mathbb{N}\} = S$. We have $\sup(S) = \sup(-S) = -\inf(S) = 1$.

4. Let S be a nonempty bounded set in $\mathbb{R}$. (3 marks)

   (a) Let $a > 0$, and let $aS := \{as : s \in S\}$. Prove that
   $$\inf(aS) = a \inf S, \quad \sup(aS) = a \sup S.$$

   (b) Let $b < 0$ and let $bS = \{bs : s \in S\}$. Prove that
   $$\inf(bS) = b \sup S, \quad \sup(bS) = b \inf S.$$

   Solution:
   (a) We will show that $a \inf S$ is a lower bound for the set aS. This will give the inequality $\inf(aS) \geq a \inf S$: fix any $s \in S$. Since $\inf S$ is a lower bound of S, we have $s \geq \inf S$ and hence $as \geq a \inf S$ for every $s \in S$.

   On the other hand, if we replace S by aS and replace a by $1/a$ in the inequality $\inf(aS) \geq a \inf S$, we obtain
   $$\inf\left(\frac{1}{a}(aS)\right) \geq \frac{1}{a}\inf(aS)$$
   i.e. $\inf S \geq \frac{1}{a}\inf(aS)$, and $a \inf S \geq \inf(aS)$. We have shown that $\inf(aS) = a \inf S$.

   Replace S by $-S$ in the equality $\inf(aS) = a \inf S$, we have $\inf(-(aS)) = a \inf(-S)$ and then $-\sup(aS) = -a \sup S$. This shows (a).

   (b) Note that $(-b) > 0$. From (a), we have $\inf(-bS) = (-b)\inf S$ and $\sup(-bS) = (-b)\sup S$. It follows that $-\sup(bS) = -b \inf S$ and $-\inf(bS) = -b \sup S$.

5. Let S be a set of nonnegative real numbers that is bounded above and let $T := \{x^2 : x \in S\}$. Prove that if $u = \sup S$, then $u^2 = \sup T$. Give an example that shows the conclusion may be false if the restriction against negative numbers is removed.

   (4 marks, 1 mark for an example)

   Solution:
   First, we show that u^2 is an upper bound of T: let $x \in S$. Since supremum is an upper bound, we have $u \geq x$. Note that $x \in S$ is nonnegative. We have $u + x \geq 0$ and hence $u^2 - x^2 = (u - x)(u + x) \geq 0$ for every $x \in S$.

   Suppose α is an upper bound of T, hence $\alpha \geq 0$. We argue that $\alpha \geq u^2$: it is trivial if $S = \{0\}$. In this case, $T = \{0\}$. Suppose $S \neq \{0\}$. There is $x \in S$ such that $x > 0$. $\alpha \geq x^2$ because α is an upper bound of T. Recall that there is a number $\sqrt{a} \geq 0$ such that $(\sqrt{a})^2 = \alpha$. We have $(\sqrt{\alpha} - x)(\sqrt{\alpha} + x) = \alpha^2 - x \geq 0$. Since $\sqrt{\alpha} + x > 0$, by multiplying its multiplicative inverse, we have $\sqrt{\alpha} - x \geq 0$. This holds for every positive element x in S, and thus $\sqrt{\alpha}$ is an upper bound of S. It follows that $\sqrt{\alpha} \geq u$. Both $\sqrt{\alpha}$ and u are nonnegative. The inequality stills holds after taking square on both sides.

   The conclusion fails if the restriction against negative numbers is removed. For example, we take $S = \{-1, 0\}$. In this case, $u = 0$ and $T = \{1, 0\}$. We have $\sup T = 1$, but $u^2 = 0$.

   The solution have been complete. For the part showing that u^2 is the least among all upper bounds of T, one may argue that $u^2 - \epsilon$ fails to be an upper bound of T for every small $\epsilon > 0$. The following can be taken as a substitute for the second paragraph: we still assume that $S \neq \{0\}$, hence $u > 0$. Note that

   $$u^2 - \epsilon \leq \left(u - \frac{\epsilon}{2u}\right)^2$$

   By the assumption $u = \sup S$, there is some $x_0 \in S$ such that $u - \dfrac{\epsilon}{2u} < x_0$. Whenever $\epsilon > 0$ is small enough, say $\epsilon < 2u^2$, the inequality preserves after taking square. This together with the inequality $u^2 - \epsilon \leq \left(u - \dfrac{\epsilon}{2u}\right)^2$ give us $u^2 - \epsilon < x_0^2$, hence $u^2 - \epsilon$ is not an upper bound of T.

3. Let $S \subseteq \mathbb{R}$ be nonempty. Prove that if a number u in $\mathbb{R}$ has the properties: (i) for every $n \in \mathbb{N}$, the number $u - 1/n$ is not an upper bound of S, and (ii) for every number $n \in \mathbb{N}$, the number $u + 1/n$ is an upper bound of S, then $u = \sup S$. (3 marks)

   Solution:
   We first show that u is an upper bound, i.e. $u \geq s$ for every $s \in S$: suppose not, there is some $s_0 \in S$ such that $s_0 > u$. By Archimedean property, there is some $n \in \mathbb{N}$ such that $1/n < s_0 - u$. Now, $u + 1/n$ is not an upper bound of S and contradicts to (ii).

   Next, we show that u is the least among all upper bounds of S: let $\epsilon > 0$, by Archimedean property, there is some $n \in \mathbb{N}$ with $1/n < \epsilon$. By property (i), $u - 1/n$ is not an upper bound of S, and hence a smaller number $u - \epsilon$ cannot be an upper bound of S. This shows that $u = \sup S$.