

MATH 2050A: Mathematical Analysis I (Appendix)
(2019 1st term)

1 Compact subsets of \mathbb{R}

Definition 1.1 Let A be a subset of \mathbb{R} . A point $z \in \mathbb{R}$ is called a limit point of A if for any $\delta > 0$, there is an element $a \in A$ such that $0 < |z - a| < \delta$. Put $D(A)$ the set of all limit points of A .

Example 1.2 (i) $D([a, b]) = D((a, b)) = [a, b]$.

(ii) $D([0, 1] \cup \{2\}) = [0, 1]$.

(iii) $D(\mathbb{N}) = \emptyset$.

(iv) $D(\{a\}) = \emptyset$ for any $a \in \mathbb{R}$.

Definition 1.3 A subset A of \mathbb{R} is said to be closed in \mathbb{R} if $D(A) \subseteq A$.

Example 1.4 (i) $\{a\}; [a, b]; [0, 1] \cup \{2\}; \mathbb{N}$ and \mathbb{R} all are closed subsets of \mathbb{R} .

(ii) (a, b) and \mathbb{Q} are not closed.

The following Lemma can be directly shown by the definition, so, the proof is omitted here.

Lemma 1.5 Let A be a subset of \mathbb{R} . The following statements are equivalent.

(i) A is closed.

(ii) For each element $x \in \mathbb{R} \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$.

(iii) If (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim x_n \in A$.

Definition 1.6 Let A be a subset of \mathbb{R} . A is said to be *compact* (more precise, *sequentially compact*) if every sequence (x_n) in A has a convergent subsequence (x_{n_k}) with $\lim_k x_{n_k} \in A$.

Example 1.7 (i) Every closed and bounded interval is compact.

In fact, if (x_n) is any sequence in a closed and bounded interval $[a, b]$, then (x_n) is bounded. Then by Bolzano-Weierstrass Theorem (see [1, Theorem 3.4.8]), (x_n) has a convergent subsequence (x_{n_k}) . Notice that since $a \leq x_{n_k} \leq b$ for all k , then $a \leq \lim_k x_{n_k} \leq b$, and thus $\lim_k x_{n_k} \in [a, b]$. Therefore A is compact.

- (ii) $(0, 1]$ is not compact. In fact, if we consider $x_n = 1/n$, then (x_n) is a sequence in $(0, 1]$ but it has no convergent subsequence with the limit sitting in $(0, 1]$.

Theorem 1.8 *A subset A of \mathbb{R} is compact if and only if A is closed and bounded.*

2 Continuous functions defined on compact sets

Throughout this section, let A be a non-empty subset of \mathbb{R} and $f : A \rightarrow \mathbb{R}$ a function defined on A .

Proposition 2.1 *Suppose that f is continuous on A . If A is compact, then there are points c and b in A such that*

$$f(c) = \max\{f(x) : x \in A\} \text{ and } f(b) = \min\{f(x) : x \in A\}.$$

Proof: By considering the function $-f$ on A , it needs to show that $f(c) = \max\{f(x) : x \in A\}$ for some $c \in A$.

We first claim that f is bounded on A , that is, there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. Suppose not. Then for each $n \in \mathbb{N}$, we can find $a_n \in A$ such that $|f(a_n)| > n$. Recall that A is compact if and only if it is closed and bounded (see Theorem 1.8). So, (a_n) is a bounded sequence in A . Then by the Bolzano-Weierstrass Theorem, there is a convergent subsequence (a_{n_k}) of (a_n) . Put $a = \lim_k a_{n_k}$. Since A is closed and f is continuous, $a \in A$, from this, it follows that $f(a) = \lim_k f(a_{n_k})$. It is absurd because $n_k < |f(a_{n_k})| \rightarrow |f(a)|$ for all k and $n_k \rightarrow \infty$. So f must be bounded. So $L := \sup\{f(x) : x \in A\}$ must exist by the Axiom of Completeness.

It remains to show that there is a point $c \in A$ such that $f(c) = L$. In fact, by the definition of supremum, there is a sequence (x_n) in A such that $\lim_n f(x_n) = L$. Then by the Bolzano-Weierstrass Theorem again, there is a convergent subsequence (x_{n_k}) of (x_n) with $\lim_k x_{n_k} \in A$. If we put $c := \lim_k x_{n_k} \in A$, then $f(c) = \lim_k f(x_{n_k}) = L$ as desired. The proof is finished. \square

Proposition 2.2 *If $f : A \rightarrow \mathbb{R}$ is continuous and A is compact, then the image $f(A)$ is compact. Furthermore, if f is injective, then the inverse map $f^{-1} : f(A) \rightarrow A$ is also continuous.*

Proof: Recall the fact that a subset of \mathbb{R} is closed if and only if it is closed and bounded (see Theorem 1.8). So, it needs to show that $f(A)$ is a closed and bounded set. We first notice that $f(A)$ is bounded by Proposition 2.1. It remains to show that $f(A)$ is a closed subset of \mathbb{R} . Let $y \in \overline{f(A)}$. Then there is a sequence (x_n) in A such that $\lim f(x_n) = y$. Then by the compactness of A , there is a convergent subsequence (x_{n_k}) of (x_n) such that $\lim_k x_{n_k} \in A$. Since f is continuous, it follows that $y = \lim_k f(x_{n_k}) = f(\lim_k x_{n_k}) \in f(A)$ and thus $f(A)$ is closed.

Concerning the last assertion, let $B = f(A)$ and $g = f^{-1} : B \rightarrow A$. Suppose that g is not continuous at some $b \in B$. Put $a = g(b) \in A$. Then there are $\eta > 0$ and a sequence (y_n) in B such that $\lim y_n = b$ but $|g(y_n) - g(b)| \geq \eta$ for all n . Let $x_n := g(y_n) \in A$. So, by the compactness of A , there is a convergent subsequence (x_{n_k}) of (x_n) such that $\lim_k x_{n_k} \in A$. Let

$a' = \lim_k x_{n_k}$. Then we have $f(a') = \lim_k f(x_{n_k}) = \lim_k y_{n_k} = b$. On the other hand, since $|g(y_n) - g(b)| \geq \eta$ for all n , we see that

$$|x_{n_k} - a| = |g(y_{n_k}) - g(b)| \geq \eta > 0$$

for all k and hence $|a' - a| > 0$. This implies that $a \neq a'$ but $f(a') = b = f(a)$. It contradicts to f being injective.

The proof is finished. \square

Remark 2.3 The assumption of the compactness in the last assertion of Proposition 2.2 is essential. For example, consider $A = [0, 1) \cup [2, 3]$ and define $f : A \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1) \\ x - 1 & \text{if } x \in [2, 3]. \end{cases}$$

Then $f(A) = [0, 2]$ and f is a continuous bijection from A onto $[0, 2]$ but $f^{-1} : [0, 2] \rightarrow A$ is not continuous at $y = 1$.

Example 2.4 By Proposition 2.2, it is impossible to find a continuous surjection from $[0, 1]$ onto $(0, 1)$ since $[0, 1]$ is compact but $(0, 1)$ is not. Thus $[0, 1]$ is not homeomorphic to $(0, 1)$.

Definition 2.5 A function $f : A \rightarrow \mathbb{R}$ is said to be uniformly continuous on A if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ with $|x - y| < \delta$.

Remark 2.6 It is clear that if f is uniformly continuous on A , then it must be continuous on A . However, the converse does not hold. For example, consider the function $f : (0, 1] \rightarrow \mathbb{R}$ defined by $f(x) := 1/x$. Then f is continuous on $(0, 1]$ but it is not uniformly continuous on $(0, 1]$. Notice that f is not uniformly continuous on A means that

there is $\varepsilon > 0$ such that for any $\delta > 0$, there are $x, y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Notice that $1/x \rightarrow \infty$ as $x \rightarrow 0+$. So if we let $\varepsilon = 1$, then for any $\delta > 0$, we choose $n \in \mathbb{N}$ such that $1/n < \delta$ and thus we have $|1/2n - 1/n| = 1/2n < \delta$ but $|f(1/n) - f(1/2n)| = n > 1 = \varepsilon$. Therefore, f is not uniformly continuous on $(0, 1]$.

Example 2.7 Let $0 < a < 1$. Define $f(x) = 1/x$ for $x \in [a, 1]$. Then f is uniformly continuous on $[a, 1]$. In fact for $x, y \in [a, 1]$, we have

$$|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x - y|}{xy} \leq \frac{|x - y|}{a^2}.$$

So for any $\varepsilon > 0$, we can take $0 < \delta < a^2\varepsilon$. Thus if $x, y \in [a, 1]$ with $|x - y| < \delta$, then we have $|f(x) - f(y)| < \varepsilon$ and hence f is uniformly continuous on $[a, 1]$.

Proposition 2.8 *If f is continuous on a compact set A , then f is uniformly continuous on A .*

Proof: Suppose that f is not uniformly continuous on A . Then there is $\varepsilon > 0$ such that for each $n = 1, 2, \dots$, we can find x_n and y_n in A with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Notice that by the compactness of A , (x_n) has a convergent subsequence (x_{n_k}) with $a := \lim_k x_{n_k} \in A$. Now applying sequentially compactness of A for the sequence (y_{n_k}) , then (y_{n_k}) contains a convergent subsequence $(y_{n_{k_j}})$ such that $b := \lim_j y_{n_{k_j}} \in A$. On the other hand, we also have $\lim_j x_{n_{k_j}} = a$. Since $|x_{n_{k_j}} - y_{n_{k_j}}| < 1/n_{k_j}$ for all j , we see that $a = b$. This implies that $\lim_j f(x_{n_{k_j}}) = f(a) = f(b) = \lim_j f(y_{n_{k_j}})$. This leads to a contradiction since we always have $|f(x_{n_{k_j}}) - f(y_{n_{k_j}})| \geq \varepsilon > 0$ for all j by the choice of x_n and y_n above. The proof is finished. \square

Proposition 2.9 *Let f be a continuous function defined on a bounded subset A of \mathbb{R} . Then the following statements are equivalent.*

(i): f is uniformly continuous on A .

(ii): There is a unique continuous function F defined on the closure \overline{A} such that $F(x) = f(x)$ for all $x \in A$.

Proof: The Part (ii) \Rightarrow (i) follows from Theorem 1.8 and Proposition 2.8 at once.

The proof of Part (i) \Rightarrow (ii) is divided by the following assertions. Assume that f is uniformly continuous on A .

Claim 1. If (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim f(x_n)$ exists.

It needs to show that $(f(x_n))$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$. Then by the uniform continuity of f on A , there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ with $|x - y| < \delta$. Notice that (x_n) is a Cauchy sequence since it is convergent. Thus, there is a positive integer N such that $|x_m - x_n| < \delta$ for all $m, n \geq N$. This implies that $|f(x_m) - f(x_n)| < \varepsilon$ for all $m, n \geq N$ and hence, **Claim 1** follows.

Claim 2. If (x_n) and (y_n) both are convergent sequences in A and $\lim x_n = \lim y_n$, then $\lim f(x_n) = \lim f(y_n)$.

By **Claim 1**, $L := \lim f(x_n)$ and $L' = \lim f(y_n)$ both exist. For any $\varepsilon > 0$, let $\delta > 0$ be found as in **Claim 1**. Since $\lim x_n = \lim y_n$, there is $N \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ for all $n \geq N$ and hence, we have $|f(x_n) - f(y_n)| < \varepsilon$ for all $n \geq N$. Taking $n \rightarrow \infty$, we see that $|L - L'| \leq \varepsilon$ for all $\varepsilon > 0$. So $L = L'$. **Claim 2** follows.

Recall that an element $x \in \overline{A}$ if and only if there is a sequence (x_n) in A converging to x .

Now for each $x \in \overline{A}$, we define

$$F(x) := \lim f(x_n)$$

if (x_n) is a sequence in A with $\lim x_n = x$. It follows from **Claim 1** and **Claim 2** that F is a well defined function defined on \overline{A} and $F(x) = f(x)$ for all $x \in A$.

So, it remains to show that F is continuous. Then F is a continuous extension of f to \overline{A} as desired.

Now suppose that F is not continuous at some point $z \in \overline{A}$. Then there is $\varepsilon > 0$ such that for any $\delta > 0$, there is $x \in \overline{A}$ satisfying $|x - z| < \delta$ but $|F(x) - F(z)| \geq \varepsilon$. Notice that for any $\delta > 0$ and if $|x - z| < \delta$ for some $x \in \overline{A}$, then we can choose a sequence (x_i) in A such that $\lim x_i = x$. Therefore, we have $|x_i - z| < \delta$ and $|f(x_i) - F(z)| \geq \varepsilon/2$ for any i large enough. Hence, for any $\delta > 0$, we can find an element $x \in A$ with $|x - z| < \delta$ but $|f(x) - F(z)| \geq \varepsilon/2$. Now consider $\delta = 1/n$ for $n = 1, 2, \dots$. This yields a sequence (x_n) in A which converges to z but $|f(x_n) - F(z)| \geq \varepsilon/2$ for all n . However, we have $\lim f(x_n) = F(z)$ by the definition of F

which leads to a contradiction. Thus F is continuous on \overline{A} .
 Finally the uniqueness of such continuous extension is clear.
 The proof is finished. □

Example 2.10 By using Proposition 2.9, the function $f(x) := \sin \frac{1}{x}$ defined on $(0, 1]$ cannot be continuously extended to the set $[0, 1]$.

3 Lipschitz functions

Definition 3.1 Let A be a non-empty subset of \mathbb{R} . A function $f : A \rightarrow \mathbb{R}$ is called a Lipschitz if there is a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. In this case. Furthermore, if we can find such $0 < C < 1$, then we call f a contraction.

It is clear that we have the following property.

Proposition 3.2 *Every Lipschitz function is uniformly continuous on its domain.*

Example 3.3 (i) : The sine function $f(x) = \sin x$ is a Lipschitz function on \mathbb{R} since we always have $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

(ii) : Define a function f on $[0, 1]$ by $f(x) = x \sin(1/x)$ for $x \in (0, 1]$ and $f(0) = 0$. Then f is continuous on $[0, 1]$ and thus f is uniformly continuous on $[0, 1]$. But notice that f is not a Lipschitz function. In fact, for any $C > 0$, if we consider $x_n = \frac{1}{2n\pi + (\pi/2)}$ and $y_n = \frac{1}{2n\pi}$, then $|f(x_n) - f(y_n)| > C|x_n - y_n|$ if and only if

$$\frac{2}{\pi} \cdot \frac{(2n\pi + \frac{\pi}{2})(2n\pi)}{2n\pi + \frac{\pi}{2}} = 4n > C.$$

Therefore, for any $C > 0$, there are $x, y \in [0, 1]$ such that $|f(x) - f(y)| > C|x - y|$ and hence f is not a Lipschitz function on $[0, 1]$.

Proposition 3.4 *Let A be a non-empty closed subset of \mathbb{R} . If $f : A \rightarrow A$ is a contraction, then there is a fixed point of f , that is, there is a point $a \in A$ such that $f(a) = a$.*

Proof: Since f is a contraction on A , there is $0 < C < 1$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. Fix $x_1 \in A$. Since $f(A) \subseteq A$, we can inductively define a sequence (x_n) in A by $x_{n+1} = f(x_n)$ for $n = 1, 2, \dots$. Notice that we have

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq C|x_n - x_{n-1}|$$

for all $n = 2, 3, \dots$. This gives

$$|x_{n+1} - x_n| \leq C^{n-1}|x_2 - x_1|$$

for $n = 2, 3, \dots$. So, for any $n, p = 1, 2, \dots$, we see that

$$|x_{n+p} - x_n| \leq \sum_{i=n}^{n+p-1} |x_{i+1} - x_i| \leq |x_2 - x_1| \sum_{i=n}^{n+p-1} C^{i-1}.$$

Since $0 < C < 1$, for any $\varepsilon > 0$, there is N such that $\sum_{i=n}^{n+p-1} C^{i-1} < \varepsilon$ for all $n \geq N$ and $p = 1, 2, \dots$. Therefore, (x_n) is a Cauchy sequence and thus the limit $a := \lim_n x_n$ exists. Since A is closed, we have $a \in A$ and hence f is continuous at a . On the other hand, since $x_{n+1} = f(x_n)$. Therefore, we have $a = f(a)$ by taking $n \rightarrow \infty$. The proof is finished. \square

Remark 3.5 The Proposition 3.4 does not hold if f is not a contraction. For example, if we consider $f(x) = x - 1$ for $x \in \mathbb{R}$, then it is clear that $|f(x) - f(y)| = |x - y|$ and f has no fixed point in \mathbb{R} .

4 Continuous functions defined on intervals

Recall that a non-empty subset I of \mathbb{R} is called an interval if it has one of the following forms.

- (i) \mathbb{R} .
- (ii) $(-\infty, a]$ or $[a, \infty)$ or $(-\infty, a)$ or (a, ∞) for some $a \in \mathbb{R}$.
- (iii) (a, b) or $(a, b]$ or $[a, b)$ or $[a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$.

Lemma 4.1 *Let I be a non-empty subset of \mathbb{R} . Suppose that there are different elements in I . Then I is an interval if and only if for any $a, b \in I$ with $a < b$, we have $[a, b] \subseteq I$.*

Proposition 4.2 (Intermediate Value Theorem): *Let I be an interval and let $f : I \rightarrow \mathbb{R}$ be a continuous function. Suppose that there are a and b in I with $f(a) < z < f(b)$. Then there is c between a and b such that $f(c) = z$.*

Proof: Notice that if we consider the function $x \in I \mapsto f(x) - z$, then we may assume that $z = 0$. Also, we may assume that $a < b$. Put $x_1 = a$ and $y_1 = b$. Now if $f(\frac{a+b}{2}) = 0$, then the result is obtained. If $f(\frac{a+b}{2}) > 0$, then we set $x_2 = a$ and $y_2 = \frac{a+b}{2}$. Similarly, if $f(\frac{a+b}{2}) < 0$, then we set $x_2 = \frac{a+b}{2}$ and $y_2 = b$. To repeat the same procedure, if there are x_N and y_N such that $f(\frac{x_N+y_N}{2}) = 0$, then the result is shown. Otherwise, we can find a decreasing sequence of closed and bounded intervals $[a, b] = [x_1, y_1] \supseteq [x_2, y_2] \supseteq \dots$ with $\lim(y_n - x_n) = 0$ and $f(x_n) < 0 < f(y_n)$ for all n . Then by the Nested Intervals Theorem, we have $\bigcap_n [x_n, y_n] = \{c\}$ for some $c \in [x_1, y_1] = [a, b] \subseteq I$ because I is an interval. Moreover, we have $\lim_n x_n = \lim_n y_n = c$. Then by the continuity of f , we see that $f(c) = \lim f(x_n) = \lim f(y_n)$. Since $f(x_n) < 0 < f(y_n)$ for all n , we have $f(c) = 0$. The proof is finished. \square

Remark 4.3 The assumption of the intervals in the Intermediate Value Theorem is essential. For example, consider $I = [0, 1) \cup (2, 3]$ and define $f : I \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1) \\ x - 1 & \text{if } x \in (2, 3]. \end{cases}$$

Then $f(0) < 1 < f(3)$ but $1 \notin f(I)$.

Corollary 4.4 *Let $f; [a, b] \rightarrow \mathbb{R}$. Suppose that $M := \sup\{f(x) : x \in [a, b]\}$ and $m = \inf\{f(x) : x \in [a, b]\}$. Then $f([a, b]) = [m, M]$.*

Proof: Notice that if $m = M$, then f is a constant function and hence, the result is clearly true.

Now suppose that $m < M$. It is clear that $f([a, b]) \subseteq [m, M]$ because $m \leq f(x) \leq M$ for all $x \in [a, b]$. For the converse inclusion, notice that since $[a, b]$ is compact, there are x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$. We may assume that $x_1 < x_2$. To apply the Intermediate Value Theorem for the restriction of f on $[x_1, x_2]$, we have $[m, M] \subseteq f([x_1, x_2]) \subseteq f([a, b])$. The proof is finished. \square

Corollary 4.5 *Let I be an interval and let $f : I \rightarrow \mathbb{R}$ be a continuous non-constant function. Then $f(I)$ is an interval.*

Proof: Notice that by Lemma 4.1, it needs to show that for any $c, d \in f(I)$ with $c < d$ implies that $[c, d] \subseteq f(I)$. Suppose that $a, b \in I$ with $a < b$ satisfy $f(a) = c$ and $f(b) = d$. Notice that $[a, b] \subseteq I$ because I is an interval. If we put $M = \sup_{x \in [a, b]} f(x)$ and $m = \inf_{x \in [a, b]} f(x)$, then by Corollary 4.4, we have

$$[c, d] \subseteq [m, M] = f([a, b]) \subseteq f(I).$$

The proof is finished. \square

Example 4.6 It is impossible to find a continuous surjection from (a, b) onto $(c, d) \cup (e, f)$ where $d \leq e$.

References

- [1] R.G. Bartle and I.D. Sherbert, Introduction to Real Analysis, (4th ed), Wiley, (2011).