MMAT 5220 Complex Analysis and Its Applications

Lecture 5

§ Cauchy integral formulas

Thm Let f be analytic at all points interior to and on a simple closed contour γ in positive orientation. Then for any $z_0 \in \mathbb{C}$ interior to γ, we have the Cauchy integral formula:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Pf: Let $\varepsilon > 0$. f is analytic at $z_0 \Rightarrow f$ is continuous at z_0.

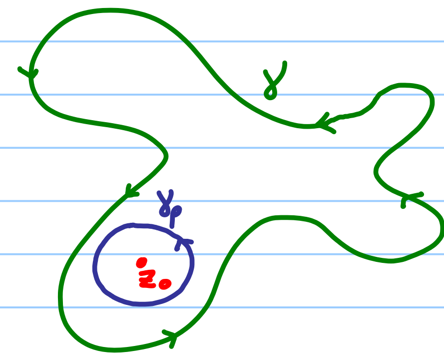
So $\exists\, \delta > 0$ s.t. $|f(z) - f(z_0)| < \varepsilon \quad \forall\, |z - z_0| < \delta$.

We then pick a sufficiently small $0 < \rho < \delta$

s.t. $B(z_0, \rho)$ is interior to γ.

Let γ_ρ be the contour:

$$\gamma_\rho(\theta) = z_0 + \rho e^{i\theta}, \quad \theta \in [0, 2\pi]$$



Cauchy-Goursat Thm implies that

$$\int_\gamma \frac{f(z)}{z - z_0}\, dz = \int_{\gamma_\rho} \frac{f(z)}{z - z_0}\, dz$$

$$= \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}}\, i\rho e^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$$

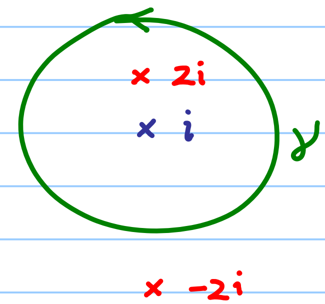
Hence

$$\left| \int_\gamma \frac{f(z)}{z - z_0}\, dz - 2\pi i f(z_0) \right| = \left| \int_0^{2\pi} \left(f(z_0 + \rho e^{i\theta}) - f(z_0) \right) d\theta \right| \leq 2\pi \varepsilon$$

Since ε is arbitrary, we obtain the formula. #

e.g. $\quad g(z) = \dfrac{1}{z^2+4}$, find $\displaystyle\int_\gamma g(z)\,dz$

where $\gamma = \{ z \in \mathbb{C} : |z - i| = 2 \}$



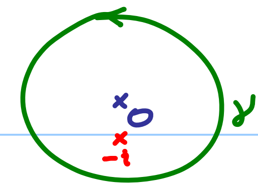
Note $\quad g(z) = \dfrac{1}{(z+2i)(z-2i)} = \dfrac{f(z)}{z-2i}$

where $f(z) = \dfrac{1}{z+2i}$ is analytic inside and on γ

So Cauchy integral formula

$$\Longrightarrow \int_\gamma g(z)\,dz = 2\pi i f(2i) = \frac{\pi}{2}$$

e.g. $\displaystyle\int_\gamma \frac{z\,dz}{(9-z^2)(z+i)}$ where $\gamma = \{ z \in \mathbb{C} : |z| = 2 \}$



$$= \int_\gamma \frac{f(z)}{z+i}\,dz \quad \text{where } f(z) = \frac{1}{9-z^2} \text{ is analytic inside and on } \gamma$$

$$= 2\pi i\, f(-i) \quad \text{by Cauchy integral formula}$$

$$= \frac{\pi}{5}$$

Thm Let f be analytic at all points interior to and on a simple closed contour γ in positive orientation. Then for any $z \in \mathbb{C}$ interior to γ and every $n \in \mathbb{N}$, we have the Cauchy integral formula:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^{n+1}}\, ds.$$

Pf : The $n=0$ case is the previous thm.

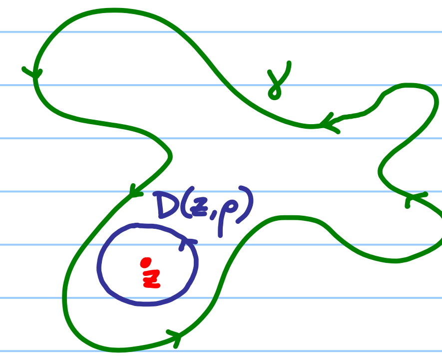
We proceed by induction on n. So we assume that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^{n+1}}\, ds \quad \& \quad f^{(n)}(z+a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z-a)^{n+1}}\, ds$$

where $|a| < \rho$ with $\rho > 0$ sufficiently small so that $B(z,\rho)$ is interior to γ.

Then $f^{(n)}(z+a) - f^{(n)}(z)$

$$= \frac{n!}{2\pi i}\int_\gamma \underbrace{\left[\frac{1}{(s-z-a)^{n+1}} - \frac{1}{(s-z)^{n+1}}\right]}_{= \, \frac{1}{(s-z-a)^{n+1}(s-z)^{n+1}}\left(\sum_{k=1}^{n+1}\binom{n+1}{k}(s-z)^{n+1-k}(-a)^k\right)} f(s)\,ds$$



$$= a\cdot\frac{(n+1)!}{2\pi i}\int_\gamma \frac{f(s)}{(s-z-a)^{n+1}(s-z)}\,ds + a^2\cdot I$$

where $$I := \frac{n!}{2\pi i}\int_\gamma \frac{\sum_{k=2}^{n+1}(-1)^k\binom{n+1}{k}(s-z)^{n+1-k}a^{k-2}}{(s-z-a)^{n+1}(s-z)^{n+1}}\,f(s)\,ds$$

Hence $\dfrac{f^{(n)}(z+a) - f^{(n)}(z)}{a} - \dfrac{(n+1)!}{2\pi i}\displaystyle\int_\gamma \frac{f(s)}{(s-z)^{n+2}}\,ds$

$$= \frac{(n+1)!}{2\pi i}\int_\gamma \Bigg[\underbrace{\frac{1}{(s-z-a)^{n+1}(s-z)} - \frac{1}{(s-z)^{n+2}}}_{= \frac{1}{(s-z-a)^{n+1}(s-z)^{n+2}}\left(\sum_{k=1}^{n+1}\binom{n+1}{k}(s-z)^{n+1-k}(-a)^k\right)}\Bigg] f(s)\,ds + a\cdot \mathrm{I}$$

$$= a(\mathrm{I} + \mathrm{II})$$

where $\mathrm{II} := \dfrac{(n+1)!}{2\pi i}\displaystyle\int_\gamma \frac{\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}(s-z)^{n+1-k}a^{k-1}}{(s-z-a)^{n+1}(s-z)^{n+2}} f(s)\,ds$

Now, as γ is compact, we can choose $R>0$ sufficiently large so that γ is contained inside $\{z\in\mathbb{C} : |z|\le R\}$ and hence

① $\cdots$ $|s-z| \leq |s|+|z| \leq 2R$

On the other hand, $B(z,\rho)$ is interior to γ, so $\forall\, s \in \gamma$, we have

② $\cdots$ $|s-z| > \rho > 0$, and

③ $\cdots$ $|s-(z+a)| \geq \text{dist}(z,\gamma) - \rho > 0.$

Let $M = \sup_{s\in\gamma} |f(s)|$ and $L =$ length of γ. Then ① + ② + ③ imply

$$|I| \leq \frac{n!}{2\pi} \int_\gamma \frac{\sum_{k=2}^{n+1} \binom{n+1}{k} |s-z|^{n+1-k} |a|^{k-2}}{|s-z-a|^{n+1} \, |s-z|^{n+1}} |f(s)|\, ds$$

$$\leq \frac{n!}{2\pi} \cdot \frac{\sum_{k=2}^{n+1} \binom{n+1}{k} (2R)^{n+1-k} \rho^{k-2}}{(\text{dist}(z,\gamma)-\rho)^{n+1} \rho^{n+1}} \cdot M \cdot L$$

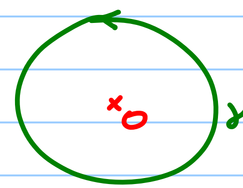
Similarly, $\quad |\mathrm{II}| \leq \frac{(n+1)!}{2\pi} \cdot \frac{\sum_{k=1}^{n+1} \binom{n+1}{k} (2R)^{n+1-k} \rho^{k-1}}{(\mathrm{dist}(z,\gamma)-\rho)^{n+1} \rho^{n+2}} \cdot M \cdot L$

Therefore, $\exists\ C = C(z, R, \rho, M, L, n) > 0$ indept of a (whenever $|a| < \rho$)

s.t. $\quad \left| \frac{f^{(n)}(z+a) - f^{(n)}(z)}{a} - \frac{(n+1)!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^{n+2}}\, ds \right| \leq |a| \cdot (|\mathrm{I}| + |\mathrm{II}|) \leq C \cdot |a|$

$\Longrightarrow \quad f^{(n+1)}(z) = \frac{(n+1)!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^{n+2}}\, ds$ by letting $a \to 0$. #

e.g. $\int_\gamma \frac{e^{2z}}{z^4}\, dz$ where $\gamma = \{ z \in \mathbb{C} : |z| = 1 \}$



$= \frac{2\pi i}{3!} \left[\left(\frac{d}{dz} \right)^3 e^{2z} \right] \Big|_{z=0} = \frac{2\pi i}{3!} \cdot 2^3 = \frac{8\pi i}{3}$.

Cor If $f(z)=u(x,y)+iv(x,y)$ is analytic at a pt $z_0=x_0+iy_0$, then f is infinitely (complex) differentiable at z_0, and $f^{(n)}$ is analytic $\forall\, n\in\mathbb{N}$. In particular, u and v are infinitely differentiable at (x_0,y_0).

Rmk Note again the sharp contrast with functions of two variables which may not even have continuous partial derivatives even if differentiable at a point.

Rmk If $f(z)=u(x,y)+iv(x,y)$ is analytic, then the Cauchy-Riemann equations imply that u, v are harmonic functions, namely, $\Delta u = 0 = \Delta v$ where $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. It is known that harmonic functions are infinitely differentiable (even real analytic).

Cor If f is continuous throughout a domain D s.t. $\int_\gamma f(z)dz = 0$ for any closed contour $\gamma \subset D$, then f is analytic in D.

Pf: Such f has an antiderivative (by a previous thm in Lecture 3) i.e. F s.t. $F'(z) = f(z)$. But then f itself is analytic. #

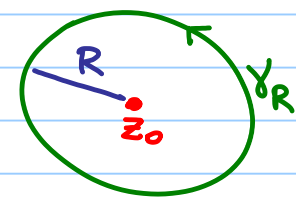
Rmk Note that f is analytic in $D \not\Leftrightarrow f$ has an antiderivative in D.
e.g. $f(z) = \frac{1}{z}$ is analytic in $\mathbb{C}\setminus\{0\}$ but has no antiderivative.
However, if D is simply-connected, then

$$f \text{ is analytic in } D \Leftrightarrow f \text{ has an antiderivative in } D$$

§ Liouville's Theorem and Fundamental Theorem of Algebra

Thm Let f be analytic on $\overline{B(z_0,R)}$, where $z_0 \in \mathbb{C}$ and $R>0$. Let $M_R = \sup_{z\in\gamma_R} |f(z)|$, where $\gamma_R = \partial B(z_0,R)$. Then we have Cauchy's inequality:

$$|f^{(n)}(z_0)| \leq \frac{n!\, M_R}{R^n}$$



Pf: By the Cauchy integral formula, we have

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \int_{\gamma_R} \frac{f(s)\,ds}{(s-z_0)^{n+1}} \right|$$

$$\leq \frac{n!}{2\pi} \, \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{n!\, M_R}{R^n} . \quad \#$$

Thm (Liouville's Thm) If f is entire (i.e. analytic throughout $\mathbb{C}$) and bounded, then f is a constant throughout $\mathbb{C}$.

Pf: Since f is bounded, $\exists M$ s.t. $|f(z)| \leq M \ \forall z \in \mathbb{C}$.

By Cauchy's inequality, we have

$$|f'(z_0)| \leq \frac{M}{R}$$

for any $z_0 \in \mathbb{C}$ and any $R > 0$. Since R can be arbitrarily large, we must have $f'(z_0) = 0 \ \forall z_0 \in \mathbb{C}$. So f is a constant function. #

The Liouville Thm has an important application:

Thm (Fundamental Thm of Algebra) Any polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ ($a_i \in \mathbb{C}$ and $a_n \neq 0$) of degree $n \geq 1$ has at least one zero, i.e. $\exists\, z_0 \in \mathbb{C}$ s.t. $P(z_0) = 0$.

Pf: We prove by contradiction. So suppose $P(z) \neq 0 \ \forall z \in \mathbb{C}$.

Then $f(z) := \dfrac{1}{P(z)}$ is an entire function.

We claim that f is bounded, which implies that f is constant by Liouville's Thm, and hence is a contradiction.

To see this, choose R large enough

s.t. $\dfrac{|a_n|}{2} \geq \left| \dfrac{a_{n-1}}{z} + \dfrac{a_{n-2}}{z^2} + \cdots + \dfrac{a_0}{z^n} \right|$ for $|z| \geq R$

$$\Rightarrow \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \geq \left| |a_n| - \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \right|$$

$$\geq \frac{|a_n|}{2} \quad \text{for } |z| \geq R.$$

So $$|f(z)| = \frac{1}{\left| z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots \frac{a_0}{z^n} \right) \right|} \leq \frac{2}{|a_n| R^n} . \quad \#$$

<u>Cor</u> A polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ ($a_i \in \mathbb{C}$ and $a_n \neq 0$) of degree $n \geq 1$ has n zeros (counted with multiplicities) in $\mathbb{C}$. More precisely, $\exists\, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ s.t. $P(z) = a_n (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$.

§ Maximum modulus principle

Cauchy integral formula implies the following mean value property:

Lemma 1 (Gauss' Mean Value Thm) If f is analytic on $\overline{B(z_0,\rho)}$, then

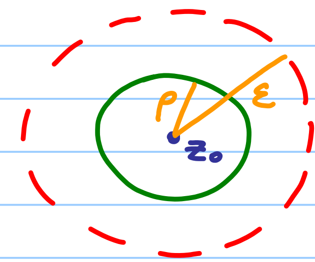
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta$$

Lemma 2 Suppose that f is analytic on $B(z_0,\varepsilon)$ and $|f(z)| \le |f(z_0)|$ $\forall z \in B(z_0,\varepsilon)$. Then $f(z) = f(z_0)$ $\forall z \in B(z_0,\varepsilon)$.

Pf: $\forall\, 0<\rho<\varepsilon$, Gauss' mean value property implies that

$$|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi}|f(z_0)|\,d\theta = |f(z_0)|$$

$$\Rightarrow \quad |f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta$$



$$\Rightarrow \quad \int_0^{2\pi} \left(|f(z_0)| - |f(z_0+\rho e^{i\theta})| \right) d\theta = 0$$

But $|f(z_0)| - |f(z_0+\rho e^{i\theta})| \geq 0 \;\; \forall\, \theta \in [0, 2\pi]$, so we must have

$$|f(z_0)| = |f(z_0+\rho e^{i\theta})| \quad \forall\, \theta \in [0,2\pi].$$

Since ρ is arbitrary, we have $|f(z_0)| = |f(z)| \;\; \forall\, z \in B(z_0, \varepsilon)$.

This implies that $f(z) = f(z_0) \;\; \forall\, z \in B(z_0,\varepsilon)$. #

Thm (Maximum Modulus Principle)

If f is analytic in a domain D and $\exists z_0 \in D$ s.t. $|f(z)| \leq |f(z_0)|$ $\forall z \in D$.

Then f is a constant function in D.

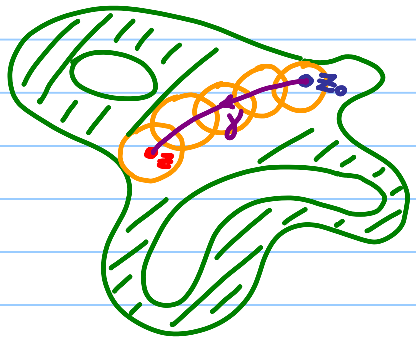
Rmk In other words, the maximum of $|f(z)|$ is attained on the boundary ∂D of D.

Pf: Let $z \in D$. Connect z_0 to z by a contour $\gamma \subset D$.

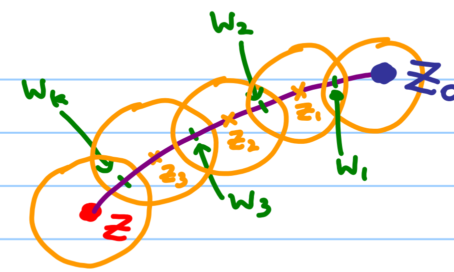
Since γ is compact, we can cover it by finitely many open disks

$$B_0 = B(z_0, \rho_0),\ B_1 = B(z_1, \rho_1),\ \dots,\ B_k = B(z, \rho_k)$$

where $B_i \subset D$, $z_i \in \gamma$ and $B_i \cap B_{i+1} \neq \emptyset$ $\forall i$.



Pick $w_{i+1} \in B_i \cap B_{i+1}$ for $i = 0, 1, \ldots, k-1$. Then



Lemma 2 applied to $B_0 \Rightarrow f(w_1) = f(z_0)$

Lemma 2 applied to $B_1 \Rightarrow f(w_2) = f(z_1) = f(w_1) = f(z_0)$

$\vdots$

Lemma 2 applied to $B_k \Rightarrow f(z) = f(w_k) = \cdots = f(w_1) = f(z_0)$. #

<u>Rmk</u> Both the mean value property and maximum principle (and hence Liouville's Thm) hold for harmonic functions.