MMAT 5220 Complex Analysis and Its Applications

Lecture 11

§ Local structure of analytic maps

Thm Let f be analytic in a domain D and $z_0 \in D$ is a zero of $f(z) - w_0$ of order $m \geq 1$. Then $\exists\, \varepsilon > 0$ and $\delta > 0$ s.t. $\forall\, 0 < |w - w_0| < \varepsilon$, $f(z) - w$ has exactly m distinct zeros in $0 < |z - z_0| < \delta$.

Pf: We choose $\delta > 0$
s.t.
- $\{z \in \mathbb{C} : |z - z_0| < \delta\} \subset D$
- $f(z) - w_0 \neq 0$ in $0 < |z - z_0| < \delta$ ($\because$ zeros are isolated)
- $f'(z) \neq 0$ in $0 < |z - z_0| < \delta$ ($\because$ f' is also analytic)

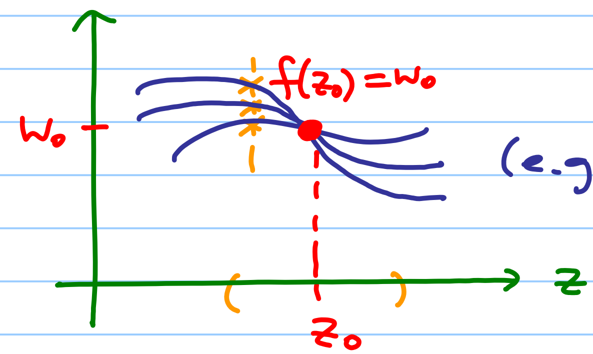
We choose $\varepsilon > 0$ s.t. $|f(z) - w_0| \geq \varepsilon$ over $|z - z_0| = \delta$.

Now for any w s.t. $0 < |w - w_0| < \varepsilon$, define $g(z) := f(z) - w$
and write $g(z) = (f(z) - w_0) + (w_0 - w) =: F(z) + G(z)$
Note that $|G(z)| = |w_0 - w| < \varepsilon \leq |F(z)|$ on $|z - z_0| = \delta$.
So Rouché's Thm implies that g has the same number of zeros
inside $|z - z_0| = \delta$ as $F(z) = f(z) - w_0$.
We conclude that $f(z) - w$ has m zeros inside $|z - z_0| = \delta$ for w
in $0 < |w - w_0| < \varepsilon$. Furthermore, $f'(z) \neq 0$ on $0 < |z - z_0| < \delta$, so
all the zeros are of order 1. #

e.g. • If f is analytic at z_0 and $f'(z_0) \neq 0$, then f is one-to-one in a nbh of z_0.

• If f is analytic at z_0, $f'(z_0)=0$ but $f''(z_0) \neq 0$, then f is 2-to-1 near z_0.

• In general, if f is analytic at z_0 and z_0 is a zero of $f(z)-f(z_0)$ of order $m \geq 1$, then f is a m-to-1 map near z_0 (e.g. $f(z)=z^m$ near 0).



(e.g. $m=3$)

Cor (Open Mapping Theorem)

If f is analytic and non-constant in a domain D, then f is open, i.e. $\forall\, w_0 = f(z_0) \in f(D)$, $\exists\, \varepsilon > 0$ s.t. $\{ w \in \mathbb{C} : |w - w_0| < \varepsilon \} \subset f(D)$.

Pf: Let $w_0 = f(z_0) \in f(D)$. Since f is non-constant, z_0 is an isolated zero of $f(z) - w_0$. The previous thm says that $\exists\, \varepsilon > 0$ and $\delta > 0$ s.t. when $0 < |w - w_0| < \varepsilon$, $f(z) - w$ has a zero over $0 < |z - z_0| < \delta$. Hence $\{ w \in \mathbb{C} : |w - w_0| < \varepsilon \} \subset f(D)$. #

Rmk The open mapping Thm can be used to deduce the maximum modulus principle.

More generally speaking, the above results determine the local structure of analytic (or holomorphic) maps between Riemann surfaces, which are natural domains of definition of multi-valued analytic functions.

(e.g. the square root function $f(z) = \sqrt{z}$ is multi-valued on $\mathbb{C}$ but well-defined and single-valued on the surface

$$S := \{ (z,w) \in \mathbb{C}^2 : w^2 = z \}$$

Namely, f is the projection map $S \longrightarrow \mathbb{C}$, $(z,w) \longmapsto w$.)

More precisely, if X and Y are compact Riemann surfaces

and if $f: X \to Y$ is a nonconstant analytic map, then it can be shown by the above results that

- f is onto and $m := \deg f \geq 1$ is well-defined.
- f is a branched covering, in particular, f is a m-to-1 map everywhere except finitely many points.
- The Riemann-Hurwitz formula holds, giving a precise relation between topology of X and that of Y.

§ Conformal mappings

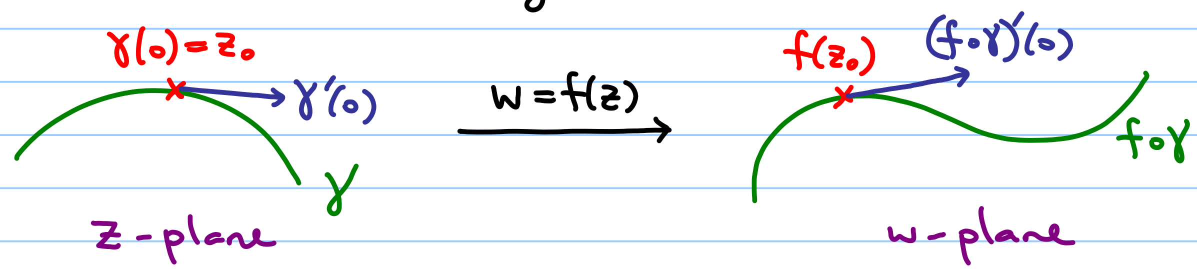
Consider a function $f : D \to \mathbb{C}$ on a domain $D \subset \mathbb{C}$.

We want to study the mapping $w = f(z)$.

Consider a path $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{C}$ defined by $\gamma(t) = x(t) + iy(t)$.

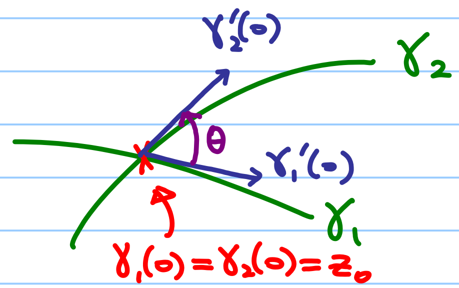
Suppose $\gamma(0) = z_0 \in D$ and f is complex differentiable at z_0.

Then $(f \circ \gamma)'(0) = f'(z_0) \cdot \gamma'(0)$ by the Chain Rule.



Note that if $f'(z_0) \neq 0$, then $\arg (f \circ \gamma)'(0) = \arg f'(z_0) + \arg \gamma'(0)$.

Now suppose we have two paths $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \mathbb{C}$ with $\gamma_1(0) = \gamma_2(0) = z_0$.



Then the (oriented) angle between γ_1 and γ_2 at z_0 is defined as the (oriented) angle between $\gamma_1'(0)$ and $\gamma_2'(0)$.

$= \arg \gamma_2'(0) - \arg \gamma_1'(0)$ (by choosing a suitable branch of arg).

Def A function f, with continuous partial derivatives, is said to be conformal at z_0 if, for any paths $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \mathbb{C}$ with $\gamma_1(0) = \gamma_2(0) = z_0$, the (oriented) angle between γ_1 and γ_2 at z_0 is equal to that between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(z_0)$.

A function is said to be conformal in a domain $D \subset \mathbb{C}$ if it is conformal at every point in D.

Thm A function f is conformal at z_0 iff f is complex differentiable at z_0 and $f'(z_0) \neq 0$.