THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Week 5 Examples

1. Assume all contours of integrals are in the counterclockwise direction. Evaluate the integrals.

   (a)
$$\int_{|z-i|=2} \frac{\mathrm{d}z}{(z^2+4)^2};$$

   (b)
$$\int_{|z|=2} \frac{\mathrm{d}z}{z^2+1};$$

   (c) Let $a \in \mathbb{C}$, $\rho > 0$, so that $\rho \neq |a|$, find
$$\int_{|z|=\rho} \frac{|\mathrm{d}z|}{|z-a|^2}.$$

Solution.

(a) Let $g(z) = \frac{1}{(z^2+4)^2}$. If we put $f(z) = \frac{1}{(z+2i)^2}$, then
$$g(z) = \frac{f(z)}{(z-2i)^2},$$
where f is analytic inside and on the circle $\{|z-i| = 2\}$.
By the Cauchy integral formula,
$$\int_{|z-i|=2} \frac{f(z)}{(z-2i)^2}\,\mathrm{d}z = \frac{2\pi i}{1} f'(2i) = 2\pi i(\frac{-2}{(4i)^3}) = \frac{\pi}{16}.$$

(b) z^2+1 has two distinct roots i and $-i$ inside the circle $\{|z| = 2\}$. By partial fraction,
$$\frac{1}{z^2+1} = \frac{1}{2i}(\frac{1}{z-i} - \frac{1}{z+i}).$$
Hence, by Cauchy integral formula,
$$\int_{|z|=2} \frac{\mathrm{d}z}{z^2+1} = \frac{1}{2i}(2\pi i - 2\pi i) = 0.$$

(c) Parametrize the contour $\{|z| = \rho\}$ by $z = \rho e^{i\theta}$ with $\theta \in [0, 2\pi]$. Then
$$\mathrm{d}z = \rho e^{i\theta} i\,\mathrm{d}\theta$$
$$|\mathrm{d}z| = \rho\,\mathrm{d}\theta = \frac{\mathrm{d}z}{ie^{i\theta}} = \rho\frac{\mathrm{d}z}{iz}$$

Therefore,

$$\begin{aligned}\int_{|z|=\rho}\frac{|\mathrm{d}z|}{|z-a|^2} &= \int_{|z|=\rho}\frac{\rho\,\mathrm{d}z}{iz(z-a)(\bar{z}-\bar{a})}\\ &= \int_{|z|=\rho}\frac{\rho\,\mathrm{d}z}{iz(z-a)(\frac{\rho^2}{z}-\bar{a})}\\ &= -i\rho\int_{|z|=\rho}\frac{\mathrm{d}z}{(z-a)(\rho^2-\bar{a}z)}\end{aligned}$$

Therefore, the integral equals $2\pi/\rho$ if $a=0$. For $a\neq 0$,

$$\int_{|z|=\rho}\frac{|\mathrm{d}z|}{|z-a|^2} = \frac{i\rho}{\bar{a}}\int_{|z|=\rho}\frac{\mathrm{d}z}{(z-a)(z-\frac{\rho^2}{\bar{a}})},$$

Notice that $|\rho^2/\bar{a}|<\rho$ if and only if $\rho<|a|$. By Cauchy integral formula, we have

$$\int_{|z|=\rho}\frac{|\mathrm{d}z|}{|z-a|^2} = \begin{cases}\frac{2\pi\rho}{\rho^2-|a|^2} & \text{if } |a|<\rho;\\ \frac{2\pi\rho}{|a|^2-\rho^2} & \text{if } \rho<|a|.\end{cases}$$

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2. If $f(z)$ is analytic for $|z|<1$ and $|f(z)|\leq 1/(1-|z|)$, find the best estimate of $|f^{(n)}(0)|$ that the Cauchy's inequality will yield.

Solution. For any $0<r<1$, if we put $M_r=\sup\{|f(z)| : z\in\partial B(0,r)\}$, then Cauchy inequality gives

$$|f^{(n)}(0)|\leq\frac{n!M_r}{r^n}$$

Moreover, $M_r\leq 1/(1-r)$ by our assumption, hence $|f^{(n)}(0)|\leq\frac{n!}{r^n(1-r)}$. For each $n\in\mathbb{N}\cup\{0\}$, we choose $r=\frac{n}{n+1}$ to minimize the bound. That is,

$$|f^{(n)}(0)|\leq (n+1)!(1+1/n)^n.$$

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3. Let f be a continuous function on $\{|z|\leq 1\}$ and analytic on $\{|z|<1\}$. If

$$|f(z)|=1 \quad \text{for every } |z|=1,$$

then f is a constant function, or $f(z)=0$ for some $|z|<1$.

Solution. By Maximum Modulus Prinicple, $|f(z)|\leq 1$ for all $|z|<1$. Suppose $f(z)\neq 0$ for all $|z|<1$. Then, $1/f(z)$ is also analytic on $|z|<1$ and $|1/f(z)|=1$ for $|z|=1$. By Maximum Modulus Principle again, $|1/f(z)|\leq 1$ for all $|z|<1$. These tell us that $|f(z)|=1$ on $\{|z|\leq 1\}$. Since the maximum of $|f(z)|$ can be attained in the interior of $\{|z|\leq 1\}$, f is a constant function. ◀