THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 5
Due Date: 16th April 2020

Compulsory Part

1. Find the residue at $z = 0$ of the following functions:

   (a) $\dfrac{1}{z + z^2}$;

   (b) $z \cos\left(\dfrac{1}{z}\right)$.

2. For each of the following functions, find all its isolated singular points, write down their principal parts, classify their types, and compute the residues:

   (a) $\dfrac{z - 1}{z^2 - 5z + 4}$;

   (b) $\sin\left(\dfrac{2}{z}\right)$;

   (c) $\dfrac{z + 1}{\cos z}$.

3. Use residues to evaluate the integral $\displaystyle\int_{|z|=3} \frac{2z - 3}{z(z + 1)} dz$.

4. Suppose that q is analytic and has a zero of order 1 at z_0. Show that $f = 1/q^2$ has a pole of order 2 at z_0 with residue given by

$$\text{Res}_{z=z_0} f(z) = -\frac{q''(z_0)}{(q'(z_0))^3}.$$

5. For any $N > 0$, let γ_N be the positively oriented boundary of the square bounded by the lines $x = \pm(N + \frac{1}{2})\pi$ and $y = \pm(N + \frac{1}{2})\pi$.

   (a) Show that

$$\int_{\gamma_N} \frac{dz}{z^2 \sin z} = 2\pi i \left(\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right).$$

   (b) Using (a), show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

   by estimating $\left| \displaystyle\int_{\gamma_N} \frac{dz}{z^2 \sin z} \right|$ in terms of N.

Optional Part

1. Find the residue at $z = 0$ of the following functions:

   (a) $\dfrac{\cot z}{z^4}$;

   (b) $\dfrac{z^3 + 2z + 1}{z^2(z+1)}$.

2. For each of the following functions, find all its isolated singular points, write down their principal parts, classify their types, and compute the residues:

   (a) $\dfrac{\sin 3z}{z}$;

   (b) $\dfrac{z^2}{2 - \sqrt{z}}$, where the principal branch if taken for $\sqrt{z}$.

3. Use residues to evaluate the integral $\displaystyle\int_{|z|=3} \frac{z^3}{4+z^2}dz$.

4. Let $a_1, a_2, \ldots, a_n$ be *distinct* complex numbers. Let γ be a circle around a_1 such that γ and its interior do not contain a_j for $j > 1$. Let $f(z) = (z - a_1)(z - a_2)\ldots(z - a_n)$. Find $\displaystyle\int_\gamma \frac{dz}{f(z)}$.