THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 4
Due Date: 2nd April 2020

Compulsory Part

1. Expand e^z into a Taylor series about the point $z = 1$.

2. Show that the Laurent series of $\dfrac{e^z}{z(z^2+1)}$ is given by

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots$$

   for $0 < |z| < 1$.

3. Find the Laurent series of $\dfrac{1}{(z-1)(z-2)}$ in

   (a) $|z| < 1$;

   (b) $1 < |z| < 2$;

   (c) $1 < |z-3| < 2$.

4. Show that the function $f(z) = 1 - \cos z$ has a zero of order 2 at $z_0 = 0$.

5. Suppose that $f(z)$ and $g(z)$ are functions analytic at z_0. If z_0 is a zero of both $f(z)$ and $g(z)$ of order $m \geq 1$, show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}.$$

Optional Part

1. With the aid of series, prove that the function f defined by

$$f(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

   is an entire function.

2. Let f be a function analytic in a domain $D \subset \mathbb{C}$ which has distinct zeros $z_1, z_2, \ldots, z_n$ of orders $m_1, m_2. \ldots, m_n$ respectively. Show that there exists an analytic function $g(z)$ on D such that

$$f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_n)^{m_n} g(z).$$

3. Let R be the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ at z_0. Show, by term-by-term differentization and mathematical induction, that

$$f^{(m)}(z) = \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n}(z - z_0)^n$$

for $|z - z_0| < R$.

4. Let f be an entire function such that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ for all $x \in \mathbb{R}$. Show that

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

for all $z \in \mathbb{C}$.