THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 3
Due Date: 19th March 2020

Compulsory Part

1. Let γ be a positively oriented circle which does not pass through $z_0 \in \mathbb{C}$. Show that

$$\int_\gamma \frac{dz}{z - z_0} = \begin{cases} 2\pi i & \text{if } z_0 \text{ lies inside } \gamma, \\ 0 & \text{if } z_0 \text{ lies outside } \gamma. \end{cases}$$

2. Let γ be the positively oriented (i.e. going in the counterclockwise direction) boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of the following integrals:

   (a) $\displaystyle\int_\gamma \frac{e^{-z}}{z - (\pi i/2)} dz$

   (b) $\displaystyle\int_\gamma \frac{\cos z}{z(z^2 + 8)} dz$

3. Let $a \in \mathbb{R}$. By integrating the function e^{az}/z around the unit circle, parametrized as $\gamma(\theta) = e^{i\theta}$, $-\pi \leq \theta \leq \pi$, show that

$$\int_0^\pi e^{a\cos\theta} \cos(a \sin\theta) d\theta = \pi.$$

4. Let $n \in \mathbb{Z}$ and γ be the positively oriented unit circle. Compute $\displaystyle\int_\gamma \frac{e^z}{z^n} dz$. (Hint: there are two cases to be considered.)

5. Let $f(z)$ be an entire function.

   (a) If $f^{(n)}(z) \equiv 0$ for some $n \in \mathbb{N}$, show that $f(z)$ is a polynomial.

   (b) Prove that if $|f(z)| < |z|^n$ for all $|z| > R$, where $R > 0$ and $n \in \mathbb{N}$, then $f(z)$ must be a polynomial. (Hint: Use the Cauchy integral formula to estimate $f^{(n+1)}(z)$.)

6. Suppose that $f(z)$ is entire and there exists $A > 0$ such that $|f(z)| \leq A|z|$ for all $z \in \mathbb{C}$. Show that $f(z) = az$ for some constant $a \in \mathbb{C}$.

Optional Part

1. Let γ be a simple closed contour in $\mathbb{C}$, $R \subset \mathbb{C}$ be the interior of γ, and f be a continuous function on γ. Show that the function

$$F(z) := \int_\gamma \frac{f(s)}{s - z} ds,$$

defined for $z \in R$, is analytic on R with

$$F'(z) = \int_\gamma \frac{f(s)}{(s-z)^2} ds$$

for $z \in R$.

2. By integrating the function

$$\frac{1}{z}\left(z + \frac{1}{z}\right)^{2n}$$

around the unit circle, parametrized as $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$, show that for any $n \in \mathbb{N}$,

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t \, dt = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

3. Suppose that $f(z)$ is entire and there exists $M \in \mathbb{R}$ such that $\mathrm{Re} f(z) \leq M$ for all $z \in \mathbb{C}$. Prove that $f(z)$ is a constant function.

4. Suppose that f is analytic in $|z| \leq R$ and there exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $|z| \leq R$. Show that, for any $n \in \mathbb{N}$, we have

$$\left|f^{(n)}(z)\right| \leq \frac{n! M R}{(R - |z|)^{n+1}}$$

for all $|z| < R$.